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Some topological properties of networks containing controlled sources

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SOME TOPOLOGICAL PROPERTIES OF NETWORKS CONTAINING CONTROLLED SOURCES

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INTRODUCTION

A large class of physical networks may be understood in terms of idealized network models containing ordinary voltage and current sources, controlled sources, and lumped linear resistors, capacitors and inductors. For a network model of arbitrary topological structure it is possible, using the concepts and notation of linear graph theory, to write a system equation which exactly describes the idealized network and approximately describes the underlying physical network. Two problems involving determinants of coefficient matrices of such system equations are the principal topics of this study. These will be referred to as the solvability problem and the topological formula problem. Since the network classes of interest are somewhat different, the two problems have in the past been approached separately using different notation and different types of linear graphs. However the problems are so closely related that understanding of one augments understanding of the other; and it was found advantageous to study the two concurrently using a unified notation and a single type of linear graph.

Solvability Problem

In order to better understand both a physical network and its idealized model, it is useful to know what conditions are necessary and sufficient to insure that the system equation has a unique solution. This is the essence of the solvability problem, which is investigated here for networks containing any number of driving sources in addition to passive elements and controlled sources.

Seshu and Reed (16) stated and proved a set of conditions necessary and sufficient to guarantee the unique solvability of the system equation for a general RLC network with imbedded voltage and current sources. They also suggested that the study be extended to networks which contain controlled sources of all types in addition to the sources and passive elements included in their theory.

The solvability studies begin with a review of Seshu and Reed's solvability theorem and the postulate upon which it is based. The postulate is then generalized to include controlled sources of all types and a number of solvability theorems are given for networks satisfying the new postulate. An unexpected result of the solvability studies is a topological formula which applies to the system determinants of transformerless high gain operational amplifier networks. This formula is shown to agree with one given recently by Sinha (17) using a less general approach.

Topological Formulas

In studying networks from the viewpoints of both analysis and synthesis, it is very useful to be able to relate the various system functions directly to substructures of the networks. A powerful tool for effecting this is the topological formula, which provides a method of evaluating network determinants directly from the network diagram rather than by algebraic expansion of the determinants themselves. This approach is useful in calculations since it eliminates the cancellations inherent in algebraically expanding determinants. It also promises to provide valuable insight in developing new methods of topological network synthesis, since this synthesis approach depends upon known relationships between the analytic functions and the topological properties of the associated network. Detailed discussions of classical and potential applications of topological formulas are given by Weinberg (19), Coates (4), and Seshu and Reed (16). Deriving such topological formulas for controlled-source networks is the second topic of this study. Since applications of topological formulas in both analysis and synthesis involve networks containing only one driving source, the networks of concern in the formula derivations constitute a proper subset of those investigated in the solvability studies.

Topological formulas for passive networks first appeared in a paper published by Kirchhoff in 1847. The subject was expanded upon by Maxwell in 1892, and then for all practical purposes forgotten, until resurrected by Percival (12) in his classical paper of 1953.

A number of investigators including Percival (11), Coates (4), and Mason (7), have derived topological formulas to describe networks which contain controlled sources. However, these formulas apply only to networks for which an element-admittance or impedance matrix exists; and thus current sources controlled by short-circuit currents and voltage sources controlled by open-circuit voltages are prohibited.

A new method of deriving topological formulas is described here which may be applied to system determinants of controlled-source networks, even when no element-admittance or impedance matrix may be defined.

The derivation procedure is demonstrated for a number of important special cases and illustrated by means of an example.

DEFINITIONS AND THEOREMS

This chapter consists of a list of definitions and theorems which will prove useful in developing the theory which follows. The definition list includes some new terms necessary for understanding the solvability studies, as well as some more familiar terms for which the definitions and notation are not entirely standardized in the literature. The theorems stated without proof may be found in the literature as indicated, whereas the theorems with proofs are believed to be original.

Definitions

1. A network element is a resistor, inductor, capacitor, voltage source or current source.
2. Associated with each network element are two real-valued functions of bounded variation of the real variable t, an element voltage and an element current.
3. If a function associated with a network element is not given as a specified function of t, it is referred to as an element variable.
4. A fixed source is an ordinary voltage or current source; that is, a 2-terminal device for which either the voltage or current is a specified function of time.
5. A controlled source is a network element for which either the voltage or the current is defined to be proportional to a current or a voltage elsewhere in the network, and for which the other element variable is completely unspecified.
6. The current or voltage of a controlled source which is proportional to some other network variable is called the controlled variable.

7. A variable to which a controlled variable is proportional is a <u>controlling variable</u>.

8. If a controlling variable may be associated with a network element, this element is called a <u>controlling element</u>.

9. A constant which specifies the ratio of a controlled variable to a controlling variable is called a <u>transmittance constant</u>.

10. The controlled sources considered here consist of eight <u>controlled source types</u>, distinguished by the nature of the controlled and controlling variables. Symbolically, the controlled source types are specified by two letters and a subscript. The first letter represents the controlled variable, the second letter the controlling variable. A "z" or a "y" subscript indicates that a finite, nonzero admittance is the controlling element for the source. A zero subscript indicates that the controlling variable is associated with either an open-circuit or a short-circuit. Thus the current-controlled current sources are denoted by $I\text{-}I_z$ or $I\text{-}I_o$, the voltage-controlled current sources by $I\text{-}V_y$ or $I\text{-}V_o$, the current-controlled voltage sources by $V\text{-}I_z$ or $V\text{-}I_o$, and the voltage-controlled voltage source by $V\text{-}V_y$ or $V\text{-}V_o$.

11. Controlled source types for which the subscript in the symbolic notation is not a zero are called <u>admittance-controlled sources</u>.

12. A network which contains one and only one controlled source is a <u>single-source network</u>.

13. A network containing n controlled sources of the same type and no other controlled sources is called an <u>n-source network</u>.

14. Networks containing admittance-controlled sources and no other controlled source types are called <u>admittance-controlled networks</u>.

15. The <u>incidence matrix</u> $A_a = [\ a_{ij}\ ]$ of a linear graph G of e edges and v vertices is a matrix of v rows and e columns, where

$a_{ij} = 1$ if edge j is incident at vertex i and oriented away from vertex i,

$a_{ij} = -1$ if edge j is incident at vertex i and oriented toward vertex i, and

$a_{ij} = 0$ if edge j is not incident at vertex i.

16. A <u>vertex matrix</u> A is a matrix of v-1 rows and e columns formed by removing a row from A_a.

The incidence and vertex matrices are defined again in chapter four using the more specialized notation necessary in the topological formula derivation. However, aside from details of notation, the definitions agree exactly with those presented here.

17. A <u>general circuit matrix</u> $B_a = [\ b_{ij}\ ]$ of G is a matrix having e columns, a number of rows equal to the number of distinct circuits in the graph, and such that

$b_{ij} = 1$ if edge j is in circuit i and the orientations of the circuit and edge coincide,

$b_{ij} = -1$ if edge j is in circuit i and the orientations do not coincide, and

$b_{ij} = 0$ if edge j is not in circuit i.

18. A <u>circuit matrix</u> B is a submatrix of B_a of e-v+1 rows and e columns, which is of rank e-v+1.

19. The <u>cut-set matrix</u> $Q_a = [\ q_{ij}\ ]$ which has one row for each cut-set of the graph and one column for each edge, is defined by:

$q_{ij} = 1$ if edge j is in cut-set i and the orientations agree,

$q_{ij} = -1$ if edge j is in cut-set i and the orientations are opposite,

and

$q_{ij} = 0$ if edge j is not in cut-set i.

20. A <u>tree</u> is a connected subgraph of a connected graph which contains all of the vertices of the graph but does not contain any circuits.

21. If G is a linear graph for a network consisting only of admittance elements, a one-to-one correspondence exists between the edges of G and the admittances of the network. When this is true, it is possible to associate with every tree of G a <u>tree product</u>, given by the product of those network admittances which correspond to the edges of the tree.

22. The <u>tree-sum</u>, Σ T, for a graph G is the sum of the tree products, where the summation is over all of the trees of the graph.

23. A <u>sub-tree</u> is a connected, circuitless subgraph which does not include all of the vertices of the graph. An isolated vertex constitutes a sub-tree with no branches.

24. A <u>2-tree</u> is a pair of unconnected sub-trees which together include all of the vertices of the graph.

25. A <u>2-tree product</u> is the product of the network admittances which correspond to the edges of a 2-tree.

26. A <u>set of 2-trees</u> of a graph is the collection of all 2-trees such that one sub-tree of each 2-tree includes one or more specified vertices and the other sub-tree of each 2-tree includes one or more different specified vertices.

27. The <u>2-tree sum</u>, $\Sigma\, T_{ab,cd}$, is the sum of the 2-tree products of all 2-trees of the graph which are such that vertices a and b are in one sub-tree and b and d are in the other sub-tree.

28. The <u>L-linkage</u> between a pair of input vertices a and b and a pair of output vertices c and d is given by

$$\Sigma\, L_{ab,cd} = \Sigma\, T_{ac,bd} - \Sigma\, T_{ad,bc}.$$

29. If an edge corresponding to an admittance p is incident to vertices c and d, the <u>K-linkage</u>, $\Sigma\, K_{ab,cd}$, of the edge p with respect to the input vertices a and b consists of a sum of signed tree products, where the summation is over all of the trees of G which contain a path from b to a through p. The sign is positive for paths bdca and negative for paths bcda. $\Sigma\, K_{ab,cd}$ is given by

$$\Sigma\, K_{ab,cd} = p(\Sigma\, L_{ab,cd}).$$

30. A <u>vertex segregation</u> of a connected graph is a classification of the vertices into two all-inclusive, mutually exclusive, nonempty sets, the X set and the NX set.

31. A <u>bridge</u> is an edge of a connected graph having one vertex in the X set and one vertex in the NX set of a vertex segregation.

32. A <u>seg</u> is the set of all bridges corresponding to any vertex segregation of a connected graph. The orientation direction of a seg is from the X set of vertices to the NX set.

Theorems

Theorem 1. There exists a linear relationship among the columns of B corresponding to the edges of a cut-set.

Proof. If the columns of the general circuit matrix B_a and the cut-set matrix Q_a of a directed graph are arranged in the same edge order, according to Seshu and Reed (16),

$$B_a Q_a^{\,T} = 0,$$

where the superscript T denotes the transpose.
This implies that

$$Q_a B_a^{\,T} = 0.$$

Since B is a subset of the rows of B_a, it is also true that

$$Q_a B^T = 0.$$

Consider cut-set r represented by row r of Q_a. On partitioning B into columns and multiplying by row r of Q_a,

$$[\, q_{r1} q_{r2} \cdots q_{re} \,] \begin{bmatrix} B_1^{\,T} \\ B_2^{\,T} \\ \cdot \\ \cdot \\ \cdot \\ B_e^{\,T} \end{bmatrix} = 0.$$

If elements $i_1, i_2, \ldots, i_k$ are cut-set r, each of $q_{ri_1}, q_{ri_2}, \ldots,$ q_{ri_k} is either +1 or -1, and all other $q_{r_s} = 0$. Hence

$$q_{ri_1} B_{i_1} + q_{ri_2} B_{i_2} + \ldots + q_{ri_k} B_{i_k} = 0,$$

which is a linear relationship among those columns of B corresponding to the edges of cut-set r. This completes the proof.

Theorem 2 (16). There exists a linear relationship among the columns of **A** which correspond to the edges of a circuit.

Theorem 3 (16). A square submatrix of **A** of order v-1 is non-singular if and only if the elements corresponding to these columns of A constitute a tree of the graph.

Theorem 4 (16). A square submatrix of B of order e-v+1 is non-singular if and only if the columns of this submatrix correspond to the chord set of some tree of the graph.

Let G be a connected linear graph with a symbol, not necessarily an admittance symbol, assigned to every edge. Then a tree t_i of G may be uniquely represented by the product π_i of the symbols which correspond to the edges of the tree. Also, the set of all trees of G may be represented by the sum of the products which correspond to the individual trees, that is by $\Sigma\, \pi$.

Theorem 5 (12). Let G_1 be a connected graph containing two distinct vertices a and b as shown in Figure 1a; and let G_2 be the graph formed by placing an edge having symbol r incident to vertices a and b, as shown in Figure 1b. Let G_3 be the graph formed by identifying vertices a and b of G_1 and then adding edge r as shown in Figure 1c. Finally, removing r from G_3 leaves the graph G_4 shown in Figure 1d. Let $\Sigma\, \pi^k$ denote the sum of products corresponding to the set of trees of graph G_k. Then

$$\Sigma\, \pi^2 = \Sigma\, \pi^1 + \Sigma\, \pi^3, \text{ or}$$

$$\Sigma\, \pi^2 = \Sigma\, \pi^1 + r\ (\Sigma\, \pi^4).$$

The second equation expresses the fact that the set of all trees of a graph which contains an edge r may be divided into two mutually exclusive sets, set u given by $\Sigma\, \pi^1$, in which no tree contains r, and set v given by $r(\Sigma\, \pi^4)$, in which every tree contains r.

If the four graphs of Figure 1 correspond to networks of admittances, then the second equation above may be written in the form

$$\Sigma\, T_2 = \Sigma\, T_1 + r(\Sigma\, T_4)$$

where r is understood in this case to be an admittance, and $\Sigma\, T_k$ denotes the tree-sum for graph k.

Theorem 6 (12). The sum of a set of L-linkages for which the output nodes form a closed circuit is zero.

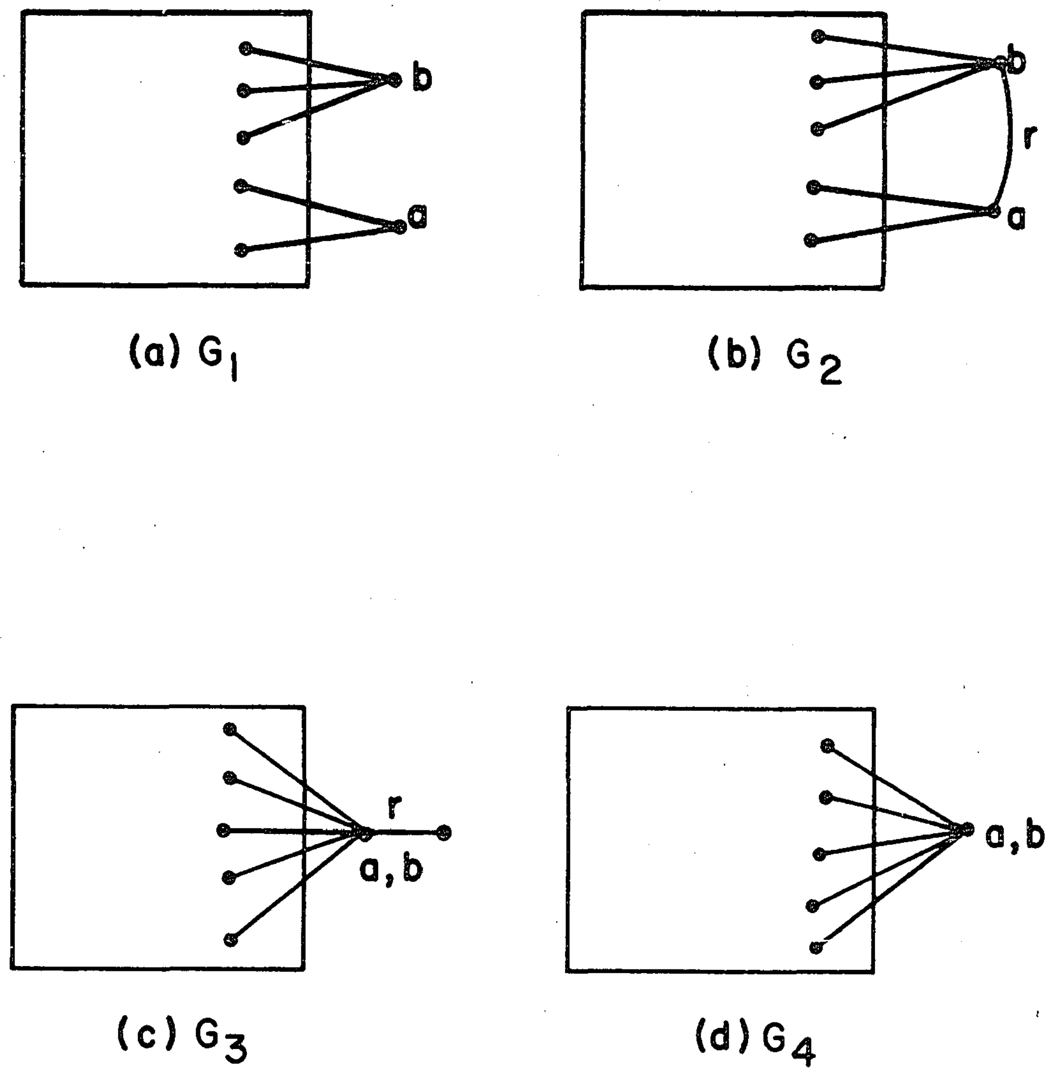
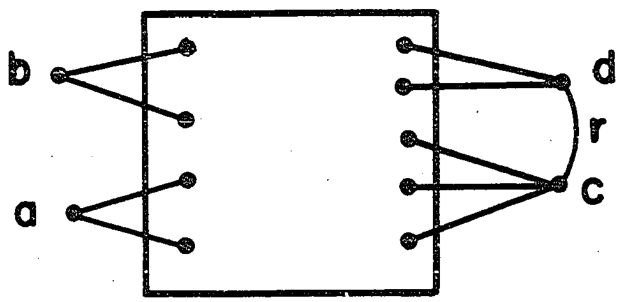
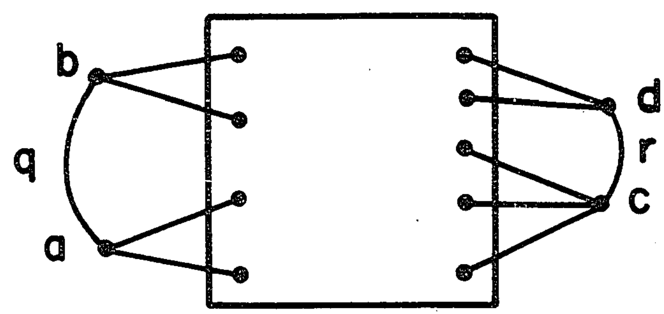


Figure 1. Graphs for Theorem 5



(a) G



(b) G'

Figure 2. Graphs illustrating Theorem 7

Theorem 7. Let G be a connected graph containing input vertices a and b and output vertices c and d as shown in Figure 2a; and let G' be the graph formed from G by adding an edge q incident to vertices a and b as in Figure 2b. Then the K-linkage, $\Sigma K_{ab,cd}$, of G and the K-linkage, $\Sigma K'_{ab,cd}$, of G' are identical.

Proof. Let ΣT represent the tree-sum of G and $\Sigma T'$ the tree-sum of G'. By Theorem 5,

$$(\Sigma T') = q(\Sigma T'') + \Sigma T,$$

where $\Sigma T''$ is the tree-sum of the graph formed from G by identifying vertices a and b. Note that every term of $q(\Sigma T'')$ contains q and that no term of ΣT contains q.

By Definition 29, each term of $\Sigma K'_{ab,cd}$ corresponds to a tree of G' which contains vertices a and b and a tree path through r. Or, except for sign, the terms of $\Sigma K'_{ab,cd}$ are a subset of the terms of $\Sigma T'$. But no tree of $\Sigma K'_{ab,cd}$ may contain edge q, for such a tree would contain a circuit, which is impossible by Definition 20. Therefore, the terms of $\Sigma K'_{ab,cd}$ must be chosen from ΣT. However the terms of both $\Sigma K'_{ab,cd}$ and $\Sigma K_{ab,cd}$ are chosen from ΣT in exactly the same manner and,since the algebraic sign associated with each term is assigned in the same way, the two K-linkages must be identical. This completes the proof.

Theorem 8 (14). The number of edges common to a seg and a circuit in a connected graph G is always even. Furthermore, the seg orientations and circuit orientations are the same in half of these common edges and opposite in half.

Theorem 9. Let G be a connected graph with vertices divided into two all-inclusive, mutually exclusive, nonempty subsets, U and V, as illustrated in Figure 3. Let a and b be vertices of V, and Let $C = c_1, c_2, \ldots, c_n$ be the set of all bridges of G. By Definition 32, C is a seg of G. Let K_{ab,u_iv_i} represent the K-linkage of the branch c_i with respect to vertices a and b. Then

$$\sum_{i}^{n} (\Sigma K_{ab,u_iv_i}) = 0.$$

Proof. By Theorem 7, the K-linkages of a new graph G' formed from G by adding an element q incident to a and b are identical to the K-linkages of G. Therefore, without loss of generality, the proof may be presented using graph G'.

By definition, K'_{ab,u_iv_i} consists of the tree products of all of the trees in G' which contain a path from b to a through the branch c_i, the sign being positive for paths bv_iu_ia and negative for paths bu_iv_ia. Now, by adding q to every tree corresponding to the set K'_{ab,u_iv_i}, we may associate with each term of the K-linkage a circuit of G'. But by Theorem 8, the number of elements common to a seg and a circuit in G' is always even. Therefore every tree corresponding to a term of K'_{ab,u_iv_i} will include a path P containing an even number of edges of C; and in traversing P from b to a, half of these edges will be traversed from the u side to the v side, and half from the v side to the u side.

For example, consider a tree corresponding to a term of K'_{ab,u_1v_1} which contains a path through c_1 from v_1 to u_1. This path also includes

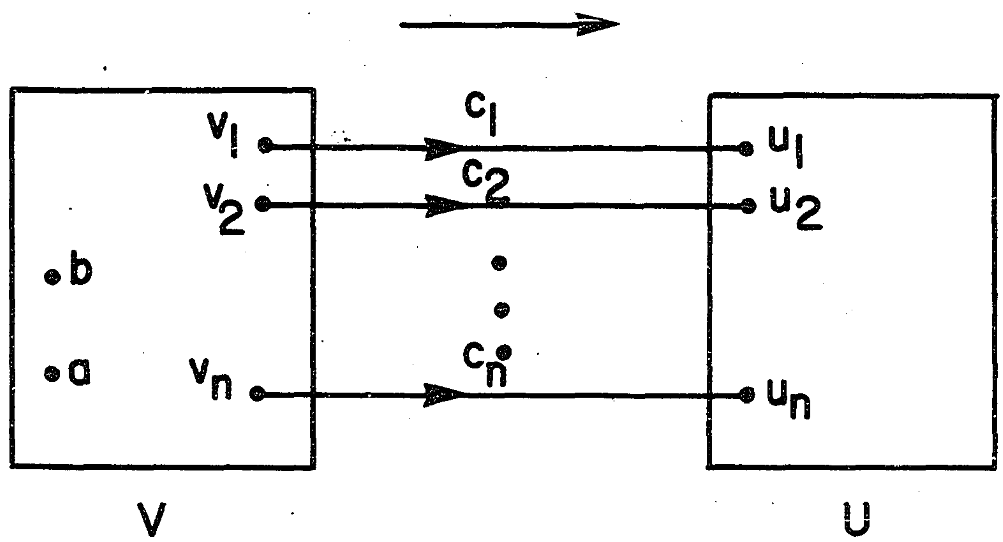


Figure 3. Graph illustrating Theorem 9

some other element of the seg, say c_2, which is traversed from u_2 to v_2. Therefore, this same tree will appear twice in the summation, once with a negative sign as a term of K'_{ab,u_2v_2}, and once with a positive sign as a term of K'_{ab,u_1v_1}. In this manner, every tree containing exactly two elements of C will appear twice in the summation with opposite signs.

Similarly, each tree containing 2k elements of C will appear k times in the sum of positive terms, and k times in the sum of negative terms, and thus cancel. Therefore, all terms in the summation cancel, and the sum of the K-linkages is zero, which completes the proof.

UNIQUE SOLVABILITY OF NETWORK EQUATIONS

The solvability theorems which constitute the major part of this chapter are patterned after Seshu and Reed's solvability theorem for networks of fixed sources and passive elements. After generalizing Seshu and Reed's basic postulate to include all eight controlled-source types, theorems for various classes of controlled-source networks satisfying this new postulate are proven using techniques similar to those used in the proof of Seshu and Reed's theorem.

Passive Networks Driven by Fixed Sources

Seshu and Reed's theory begins with a directed linear graph G which is constructed and labeled such that there is a one-to-one correspondence between the edges of G and the elements of the network diagram, including the sources, and a one-to-one correspondence between the vertices of G and the nodes of the network diagram. The edge orientations of the graph, as usual, agree with the assumed directions of current flow in the network elements. With A and B denoting the vertex and circuit matrices respectively for this linear graph, Kirchhoff's current and voltage laws may be written in the general forms

$$Ai(t) = 0 \tag{1}$$

and

$$Bv(t) = 0, \tag{2}$$

where $i(t)$ is a column matrix of element currents of the network and $v(t)$ is a column matrix of element voltages. Seshu and Reed regard Equations

1 and 2 as postulates which describe those relations between the element variables which are a consequence of the interconnection of the elements into a network structure. The nature of the individual elements, whether interconnected or not, requires that another set of relations between the variables be satisfied. A postulate, hereafter denoted by P, defines the nature of these relationships, or in effect specifies the nature of the elements which may be included in the network model described by the theory. Seshu and Reed's Postulate P may be stated in the following manner.

Postulate P. The current and voltage functions associated with each element of a network are required to satisfy a system of integro-differential equations of the general form

$$v(t) = L \frac{di(t)}{dt} + Ri(t) + D \int_0^t i(x)\, dx + e(t) + v_c(0+) \qquad (3)$$

where R and D are real, diagonal matrices with nonnegative entries on the main diagonal, and L is real, symmetric with the nonzero rows and columns of L constitutes a positive definite submatrix.

The restriction on the L matrix is equivalent to prohibiting perfectly coupled transformers. The matrix e(t) provides a set of alternative symbols for the voltage of the fixed sources of the network, so that the column matrices v(t) and i(t) in Equation 3 may include all of the element variables.

The three equations are written more concisely in Laplace transform notation as

$$AI(s) = 0, \tag{4}$$

$$BV(s) = 0, \tag{5}$$

and

$$V(s) = E(s) + Z(s)I(s) + \frac{1}{s} v_c\ (0+) - Li_L(0+), \tag{6}$$

where $\frac{1}{s} v_c(0+)$ and $Li_L(0+)$ are the transforms of the initial capacitor voltages and inductor currents respectively.

Seshu and Reed state that the rows and columns in $Z(s)$ which correspond to the sources are zero, and that whenever a row and column of $Z(s)$ is zero the corresponding element is a source. Thus, in effect, Postulate P states that every network element may be placed into one of two mutually exclusive sets. Each member of the first set is a fixed source for which either the element voltage or the element current is a known function. Since no function relating the voltage and the current is specified for such elements, the corresponding rows and columns of $Z(s)$ include only zeros. Each member of the second set is passive element such that the voltage vector, $V_p(s)$, and the current vector, $I_p(s)$, corresponding to the collection of all such elements satisfy an equation

$$V_p(s) = Z_p(s) + \frac{1}{s} \bar{v}_c\ (0+) - \bar{L}\bar{i}_L\ (0+), \tag{7}$$

which may be obtained from Equation 6 by (a) omitting $E(s)$, (b) omitting every row of $V(s)$, $I(s)$, $v_c(0+)$ and $i_L(0+)$ which corresponds to a voltage or current source, and (c) omitting every row and column of $Z(s)$ and L which corresponds to a voltage or a current source. It follows that $Z_p(s)$

is a symmetric matrix with a nonzero entry in every row and in every column. Furthermore, the order of $Z_p(s)$ is equal to the number of passive elements contained in the network.

For a connected network of e elements, consisting of h current sources, k voltage sources, and e-(h+k) passive elements, AI(s) and BV(s) together provide a set of e equations. After eliminating the h+k trivial equations from Equation 6, there remains the set of e-(h+k) equations given by Equation 7. Therefore a total of 2e-(h+k) simultaneous equations are available to describe the network. Since both a voltage and a current are associated with each network element, 2e quantities describe the network. However, h+k currents and voltages are known functions, leaving a total of 2e-(h+k) unknown variables to be determined, which agrees with the number of equations available. The conditions required to insure that the equations are linearly independent are given in Seshu and Reed's solvability theorem, which may be stated in the following form.

Theorem 10. For a connected network, the equations

$$
\begin{aligned}
&AI(s) = 0 \\
&BV(s) = 0 \qquad\qquad (8) \\
&V(s) = E(s) + Z(s)I(s) + \frac{1}{s} v_c\,(0+) - Li_L\,(0+)
\end{aligned}
$$

have a unique solution for V(s) and I(s) if and only if

(a) there is no circuit consisting only of voltage sources, and

(b) there is no cut-set consisting only of current sources.

The assumption of connectedness involves no loss of generality, for with no change in any of the element voltages or currents, it is always possible to connect a node of each separate part to a ground node with the only effect being that the common nodes are constrained to have the same potential.

For purposes of comparison with the solvability conditions to be derived for more general networks, it should be pointed out that the solvability of the system of Equations 8 depends upon the topological locations of the sources and upon nothing else.

Seshu and Reed's proof does not establish that the functions obtained by solving the system of Equations 8 for the element variables are themselves Laplace transforms. Nor does it prove that the corresponding time functions, if they may be found, satisfy the initial conditions. In fact, Seshu and Reed state that these existence theorems remain as unsolved problems. The existence theorems are similarly avoided in the following study of networks containing controlled sources.

Networks Containing Controlled Sources

When controlled sources of all types are admitted in addition to the element types considered by Seshu and Reed, it is expedient to make some modifications in the network diagram before beginning to write the network equations.

Short-circuit currents, which ordinarily would not appear as variables in the network equations, must appear explicitly when they are the controlling currents of sources. Likewise, open-circuit voltages which control sources must appear explicitly in the equations. These situations may be treated systematically in the matrix equations by following a suggestion of Reed and Reed (13). This suggestion is to replace those short-circuits which carry controlling currents by voltage sources having output functions specified to be zero, and those open-circuits having terminal voltages serving as controlling variables by current sources with current specified to be zero. The functional role of these additional sources is emphasized by referring to them as controlling elements.

The end result of adding a source in this manner is to add one equation and one unknown variable to the original system of equations. Each source increases the number of network elements e by one, and thus adds exactly one Kirchhoff's law equation to the system, since the number of such equations is given by e. The specified output function for such a source appears on the right side of the network equations with the other known time functions, while the unspecified variable appears on the left, increasing the number of unknowns by one.

After the network diagram has been modified to include open and short-circuit controlling elements whenever necessary, a directed linear graph is constructed and labeled exactly as in Seshu and Reed's theory. That is, the graph is such that there is a one-to-one correspondence between the vertices and edges of the graph and the nodes and elements of the network respectively, with each individual controlled source, fixed source, and passive element represented by an edge of the graph. It is now possible to formulate a set of postulates for controlled-source networks.

Network postulates

The first two postulates for networks containing controlled sources are given by Equations 1 and 2. However A and B now represent the vertex and circuit matrices of the linear graph for the more general network, and $v(t)$ and $i(t)$ include all of the element variables of the modified network. Again, a third postulate is necessary to precisely describe the types of network elements to be included in the theory. This postulate, hereafter referred to as P', is stated as follows.

Postulate P'. The vector of current and voltage functions associated with the elements of a network may be written in the partitioned form

$$\begin{bmatrix} v(t) \\ i(t) \end{bmatrix} = \begin{bmatrix} v_1(t) \\ v_2(t) \\ i_1(t) \\ i_2(t) \end{bmatrix} , \tag{9}$$

where $v_1(t)$ and $i_1(t)$ correspond to the variables associated with the fixed sources and the passive network elements, and $v_2(t)$ and $i_2(t)$ correspond to the variables associated with the controlled sources. The elements of $v_1(t)$ and $i_1(t)$ are required to satisfy the system of integro-differential equations given by Equation 3. That is, $v_1(t)$ and $i_1(t)$ satisfy Postulate P. Also, the controlled source variables, $v_2(t)$ and $i_2(t)$, are required to satisfy an equation of the form

$$\begin{bmatrix} v_2(t) \\ i_2(t) \end{bmatrix} = C \begin{bmatrix} v_1(t) \\ i_1(t) \end{bmatrix} , \tag{10}$$

where:

(a) Every nonzero element of C is a positive real number.

(b) There exists a permutation of the rows and columns of C which changes C to the form

$$\begin{bmatrix} \mathcal{A} & 0 \\ 0 & 0 \end{bmatrix},$$

where $\mathcal{A}$ is a diagonal matrix with order equal to the number of controlled sources in the network and with no two rows or columns of corresponding to the same network element.

The nonzero elements of $\mathcal{A}$ are the transmittance constants for the controlled sources. The requirement that these constants be positive numbers is no restriction, since reversing the assumed polarity of a controlled source relative to that of the controlling element for that source is equivalent to changing the sign of the transmittance constant. Rows of $\mathcal{A}$ corresponding to subsets of the variables of $v_2(t)$ and $i_2(t)$ correspond to controlled voltage and current sources respectively. The columns of $\mathcal{A}$ correspond to the controlling variables, no two of which may be associated with the same network element. Equation 10 thus requires a one-to-one correspondence between not only the controlled and controlling variables, but also between the controlled and controlling elements.

The class of networks to which the theory applies is considerably more general than is immediately obvious from the last statement. Voltage and current sources with output specified to be zero may be added to the original diagram to accommodate networks in which every controlling element is not associated with a unique controlled source. Also, voltage sources in series or current sources in parallel may be used for networks in which every controlled source is not associated with a unique controlling element.

Many properties of controlled-source networks may be understood by thinking of the controlled sources as being imbedded in networks of elements satisfying Seshu and Reed's Postulate P. To exploit this viewpoint, and to simplify the proofs, it is useful to define two networks, N_2 and N_3, which are closely related to the general controlled-source network, N_1, which is the subject of this study.

Related networks useful in solvability studies

Network N_1 consists of a collection of elements satisfying Postulate P', interconnected in a manner which is completely general within the usual assumption of connectedness. Initial conditions are assumed to be associated with the capacitors and inductors.

Network N_2 is formed from N_1 by replacing every controlled source by a fixed source of the same type and polarity, called a replacement source. That is, corresponding to each controlled current source in N_1 is a fixed current source in N_2 occupying the same topological position in the network. Similarly, corresponding to every controlled voltage source in N_1 is a fixed voltage source in N_2. Initial conditions of N_2 are identical to those of N_1.

Let S_a denote the set of all current sources, both fixed and controlled, in N_1 and let S_b denote the set of all voltage sources, both fixed and controlled, in N_1. The following properties of N_2 follow directly from its definition. (a) If N_1 contains no cut-set consisting entirely of elements from S_a, then N_2 contains no cut-set consisting entirely of current sources. (b) If N_1 contains no circuit consisting entirely of elements from S_b, then N_2 contains no circuit consisting only of voltage sources. (c) If N_1 is connected, N_2 is also connected. (d) The elements of N_2 satisfy Postulate P.

N_3 is a network derived from N_2 by specifying that the time function of every fixed source which is not a replacement source be zero, and by setting all initial conditions to zero. The elements of N_3 which correspond to controlling elements of N_1 will, for convenience, be referred to as controlling elements, even though N_3 itself contains no controlled sources. In applying the results to actual problems, it will be useful to note that specifying zero time functions for sources is equivalent to properly removing these sources from the network, in the sense that the values of all of the network variables are the same in either case.

Beginning with Kirchhoff's laws, Postulate P', and the definitions of N_2 and N_3, it is possible to prove a number of theorems concerning the unique solvability of network equations for controlled-source networks. Admittance-controlled single-source networks are treated first and in considerable detail because they provide a basis for understanding the more general networks. Many of the statements made in the discussion of networks containing a single I-V_y source are applicable with only minor changes to the other single-source networks, but in the interest of conciseness, are not repeated. From single-source networks, the discussion advances to general admittance-controlled networks, and then to networks containing sources controlled by short and open-circuit variables.

Finally, several aspects of controlled-source networks in which the values of the transmittance constants are assumed to approach infinity are discussed in detail.

Admittance-controlled single-source networks

Figure 4 illustrates the form of the networks N_1, N_2, and N_3 when N_1 is a connected network of v nodes and e elements containing a single I-V_y source. Since the network obeys Kirchhoff's laws and Postulate P', the transformed equations describing N_1 may be written in the partitioned form

$$\left[\begin{array}{c|ccccccc} -1/\alpha & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline A_c & A_{p1} & A_{p2} & A_E & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_c & B_{p1} & B_{p2} & B_J \\ 0 & z_{p1} & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & Z_{p2} & 0 & 0 & 0 & -U & 0 \end{array}\right] \left[\begin{array}{c} I_c \\ \hline I_{p1} \\ I_{p2} \\ I_E \\ V_c \\ V_{p1} \\ V_{p2} \\ V_J \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline -A_J J \\ -B_E E \\ I_{n1} \\ I_{n2} \end{array}\right], \tag{11}$$

where:

α is the transmittance constant of the controlled source.

$[\, A_c \; A_{p1} \; A_{p2} \; A_E \; A_J \,]$ is a vertex matrix for the graph G_1 of N_1, with any vertex chosen as reference.

$[\, B_c \; B_{p1} \; B_{p2} \; B_E \; B_J \,]$ is a circuit matrix of G_1.

z_{p1} is the impedance of the controlling element.

Z_{p2} is the element impedance matrix for the other passive elements of N_1.

U is an identity matrix.

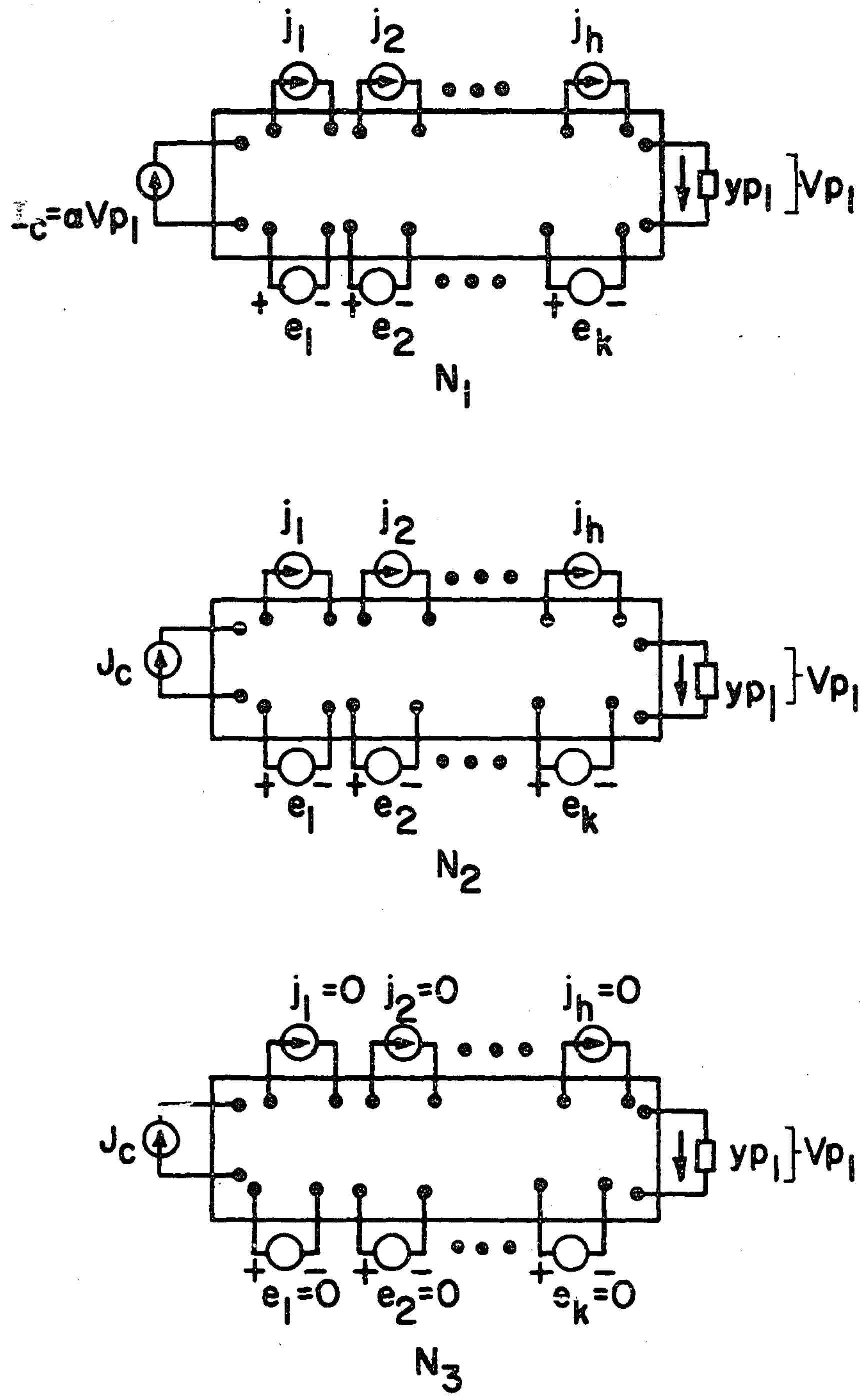


Figure 4. N_1, N_2, and N_3 for an I-V_y network

I_c and V_{p1} are the controlled and controlling variables respectively.

V_c and I_{p1} are the voltage and current of the controlled and controlling elements, respectively.

I_{p2} and V_{p2} are current and voltage vectors for the passive elements, excluding z_{p1}.

I_E and V_J are current and voltage vectors for the fixed sources.

J and E are matrices of the current and voltage driving functions of N_1.

I_{n1} and I_{n2} are the initial conditions for N_1.

Theorem 11. Equation 11 has a unique solution if and only if:

1. N_1 contains no cut-set consisting only of elements belonging to S_a.

2. N_1 contains no circuit consisting only of elements belonging to S_b.

3. $\frac{1}{\alpha} \neq t_{p1}$,
where t_{p1} is defined by

$$t_{p1} = \frac{V_{p1}}{J_c}$$

for N_3, with J_c denoting the replacement current of N_2 and N_3.

Proof. If Equation 11 has a unique solution, every subset of the columns of the coefficient matrix must be linearly independent, and in particular, the columns containing B_c and B_J must be linearly independent. But, by Theorem 1, there exists a linear relation among the colums of B corresponding to a cut-set. Since the columns of B_c and B_J correspond to the elements of S_a, no subset of elements of S_a may constitute a cut-set of N_1. Therefore condition 1 is necessary.

By the same reasoning, the columns containing A_E must be independent. But by Theorem 2, there exists a linear relation among the columns of A corresponding to the edges of a circuit. Since the columns of A_E correspond to the elements of S_b, condition 2 is necessary.

Since N_1 is connected, N_2 is a connected network satisfying Kirchhoff's laws and Postulate P. Furthermore, conditions 1 and 2, just established, insure that N_2 contains no cut-set of current sources and no circuit of voltage sources. Therefore, by Theorem 10 the system of equations for N_2 has a unique solution.

A more concise notation for use in the remainder of the proof is introduced by rewriting Equation 11 in the form

$$\left[\begin{array}{c|c} m_{11} & m_{12} \\ \hline m_{21} & M \end{array}\right] \left[\begin{array}{c} I_c \\ \hline N \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline D \end{array}\right] , \tag{12}$$

where the dashed lines of Equations 11 and 12 serve to define the submatrices of Equation 12 in terms of the submatrices of Equation 11. In N_2, a fixed source having one of its time functions specified replaces the controlled source of N_1 for which neither time function is known. Therefore, the number of equations required to describe N_2 is one less than the number required to describe N_1. In fact, again using J_c to denote the replacement current of N_2, the network equation for N_2 may be written in the notation of Equation 12 as

$$MN = D - m_{21} J_c . \tag{13}$$

Equation 13 is the form taken by the system of Equation 8 for N_2 after all known quantities have been placed on the right and the trivial equations have been eliminated.

Since Equation 13 has a unique solution, M^{-1} exists, and the solution of Equation 13 may be written in the form

$$N = M^{-1} D - M^{-1} m_{21} J_c \qquad (14)$$

which will be useful later.

Of more immediate interest is a nonsingular transformation matrix, T_1, defined by

$$T_1 = \begin{bmatrix} 1 & 0 \\ & \\ 0 & M^{-1} \end{bmatrix} . \qquad (15)$$

The partitioning of T_1 is such that it may be used to transform Equation 12 into the equivalent equation

$$\begin{bmatrix} m_{11} & m_{12} \\ & \\ M^{-1} m_{21} & U \end{bmatrix} \begin{bmatrix} I_c \\ \\ N \end{bmatrix} = \begin{bmatrix} 0 \\ \\ M^{-1} D \end{bmatrix} \qquad (16)$$

which, according to Hohn (5), has the same solution as Equation 12. If the original system Equation 11 has a unique solution, then it is necessary that Equation 16 also have a unique solution. The next step is to recognize and interpret the meaning of the submatrices of Equation 16.

The term $M^{-1}D$ of Equation 16 is recognized from Equation 14 as the solution vector for N_2 when J_c equals zero. However, this is, by definition of N_2, the solution vector for N_1 when the controlled source is replaced by a current source having a time function of zero. This is equivalent to stating that $M^{-1}D$ is the solution vector which would be obtained for N_1 if the controlled current source were properly removed.

In Equation 14, the term $-M^{-1}m_{21}J_c$ is recognized as the solution vector of N_2 for the special case in which D = 0. Every source in N_2 except J_c then has a time function of zero and the initial conditions are zero. But, by definition, this is the vector of solutions for N_3. This means that the term $M^{-1}m_{21}$ which appears in Equation 16 is a column matrix with each term representing the negative of the transfer function of N_3 which relates one of the variables to J_c.

Now to clarify the results, Equation 16 is rewritten in the expanded form

$$\begin{bmatrix} -1/\alpha & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -f_{p1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -f_{p2} & 0 & U & 0 & 0 & 0 & 0 & 0 \\ -f_E & 0 & 0 & U & 0 & 0 & 0 & 0 \\ -t_c & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -t_{p1} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -t_{p2} & 0 & 0 & 0 & 0 & 0 & U & 0 \\ -t_J & 0 & 0 & 0 & 0 & 0 & 0 & U \end{bmatrix} \begin{bmatrix} I_c \\ I_{p1} \\ I_{p2} \\ I_E \\ V_c \\ V_{p1} \\ V_{p2} \\ V_J \end{bmatrix} = \begin{bmatrix} 0 \\ I_{p10} \\ I_{p20} \\ I_{E0} \\ V_{c0} \\ V_{p10} \\ V_{p20} \\ V_{J0} \end{bmatrix} \qquad (17)$$

where:

The f's are matrices of transfer functions relating the currents of N_3 to J_c.

The t's are matrices of transfer functions relating the voltages of N_3 to J_c.

The nonzero terms on the right are vectors of values the variables of N_1 would assume if the controlled source were properly removed from N_1.

After subtracting the row containing t_{p1} from the first row, Equation 17 becomes

$$\begin{bmatrix} (t_{p1}-1/\alpha) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -f_{p1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -f_{p2} & 0 & U & 0 & 0 & 0 & 0 & 0 \\ -f_{E} & 0 & 0 & U & 0 & 0 & 0 & 0 \\ -t_{c} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -t_{p1} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -t_{p2} & 0 & 0 & 0 & 0 & 0 & U & 0 \\ -t_{J} & 0 & 0 & 0 & 0 & 0 & 0 & U \end{bmatrix} \begin{bmatrix} I_c \\ I_{p1} \\ I_{p2} \\ I_E \\ V_c \\ V_{p1} \\ V_{p2} \\ V_J \end{bmatrix} = \begin{bmatrix} -V_{p10} \\ I_{p10} \\ I_{p20} \\ I_{E0} \\ V_{c0} \\ V_{p10} \\ V_{p20} \\ V_{J0} \end{bmatrix} \qquad (18)$$

Now if Equation 11 has a unique solution, Equation 18 has a unique solution, since the two are related by a nonsingular transformation. But Equation 18 has a unique solution only if $(t_{p1} - 1/\alpha)$ does not vanish, which proves that the third condition of the theorem is necessary.

To see that conditions 1, 2, and 3 are also sufficient, it is only necessary to note that conditions 1 and 2 imply that matrix M of Equation

15, is nonsingular, establishing the equivalence of Equations 11 and 18. But the nonsingularity of the coefficient matrix of Equation 18, which depends entirely upon the term $(t_{p1} - 1/\alpha)$, is a sufficient condition for the unique solvability of Equation 18, and therefore of Equation 11. This completes the proof.

In Equation 11, the impedance matrix appears in the form

$$\begin{bmatrix} z_{p1} & 0 \\ 0 & z_{p2} \end{bmatrix},$$

tacitly implying by the absence of explicit off-diagonal terms that the controlling element is not an inductor which is magnetically coupled to another inductor. The equations are written in this form to simplify the mathematical expressions. The dashed lines in the coefficient matrix of Equation 11 show that the impedance matrix in question is a submatrix of M of Equation 12. And M, upon which the proof depends, is nonsingular even when the controlling element of N_1 has magnetic coupling, since N_2 still satisfies Postulate P in this case. Therefore the proof is valid even when the controlling element is a magnetically coupled inductor.

It is interesting to note that conditions 1 and 2 of Theorem 11, like conditions a and b of Theorem 10, are location constraints for the sources, and depend only upon the topological arrangement of the network elements. This was not completely unexpected, since controlled and fixed current sources share the important property of having one variable, the element voltage, which may be found only by means of Kirchhoff's voltage law.

Equation 17 shows the nature of the special difficulty which may arise when a network contains a controlled source. There is a possibility that the controlled source transmittance constraint, the first row of Equation 17, may duplicate in the coefficient matrix a constraint imposed between the same two variables by the network in which the source is imbedded. The row containing t_{p1} expresses this second constraint. When this occurs, the two rows of the coefficient matrix are linearly dependent, in fact identical, and the network equation has no unique solution.

It follows that for Equation 11 to have a unique solution, it is necessary that $(t_{p1} - 1/\alpha)$ not vanish identically in the Laplace-transform variable s. Ordinarily, t_{p1} is a function of s; and α is real by Postulate P'. When t_{p1} is a function of s, the network equation has a unique solution provided conditions 1 and 2 are also satisfied.

Difficulties may arise for resistive networks, which have transfer functions that are real numbers. The transfer function t_{p1} may also be a real number for networks containing reactive elements, as illustrated by the example shown in Figure 5. Figure 5a shows the controlled-source network N_1. Figure 5b shows a network equivalent to the corresponding N_3. The transfer function for this network is seen to be simply $t_{p1} = r$. This raises the question, "Under what conditions does a general 2-port have transfer functions which are real numbers?" This matter is discussed in more detail in the section on new research problems.

When t_{p1} is a real number and conditions 1 and 2 are satisfied, the remaining solvability condition is readily visualized in terms of a pair of straight lines as shown in Figure 6. One line represents the constraint imposed upon the controlling and controlled variables by the transmittance constant α , while the other represents the constraint imposed upon the same two variables by the network in which the controlled source is embedded. Intersection of the two lines implies that the network equations have a unique solution, with the point of intersection indicating the values of two of the variables. Since α must be positive to satisfy Postulate P', negative t_{pl} guarantees that a unique solution exists.

If the two lines are parallel, but not colinear, the network equations are inconsistent. Figure 6 suggests that this occurs when $t_{pl} = 1/\alpha$ and $V_{plo} \neq 0$. Equation 18 shows that these conditions make the ranks of the coefficient and augmented matrices of the transformed equations unequal. The equations for the network of Figure 5 are inconsistent when $1/\alpha = r$ and $J_1 + J_2 \neq 0$, for in this case $V_{plo} = (J_1 + J_2)\ r$.

If the constraints are such that the lines are parallel and colinear, the network equations have an infinite number of solutions, any pair of values $(I_c, 1/\alpha\ I_c)$ being acceptable for the controlled and controlling variables respectively. Figure 6 shows that t_{pl} must equal $1/\alpha$ and V_{plo} must be zero for the network equations to have non-unique solutions. Equation 18 indicates that those variables related to J_c of N_3 by a transfer function which is identically zero may be determined uniquely, whereas every variable which is not isolated from the replacement source in this manner will have a solution which depends upon I_c. The network of Figure 5 has an infinite number of solutions when $J_1 = -J_2$ and $1/\alpha = r$. Also, in this case, the voltage and current of L_3 have unique values, whereas the other variables depend upon I_c.

When network N_1 contains no transformers, the transfer functions of

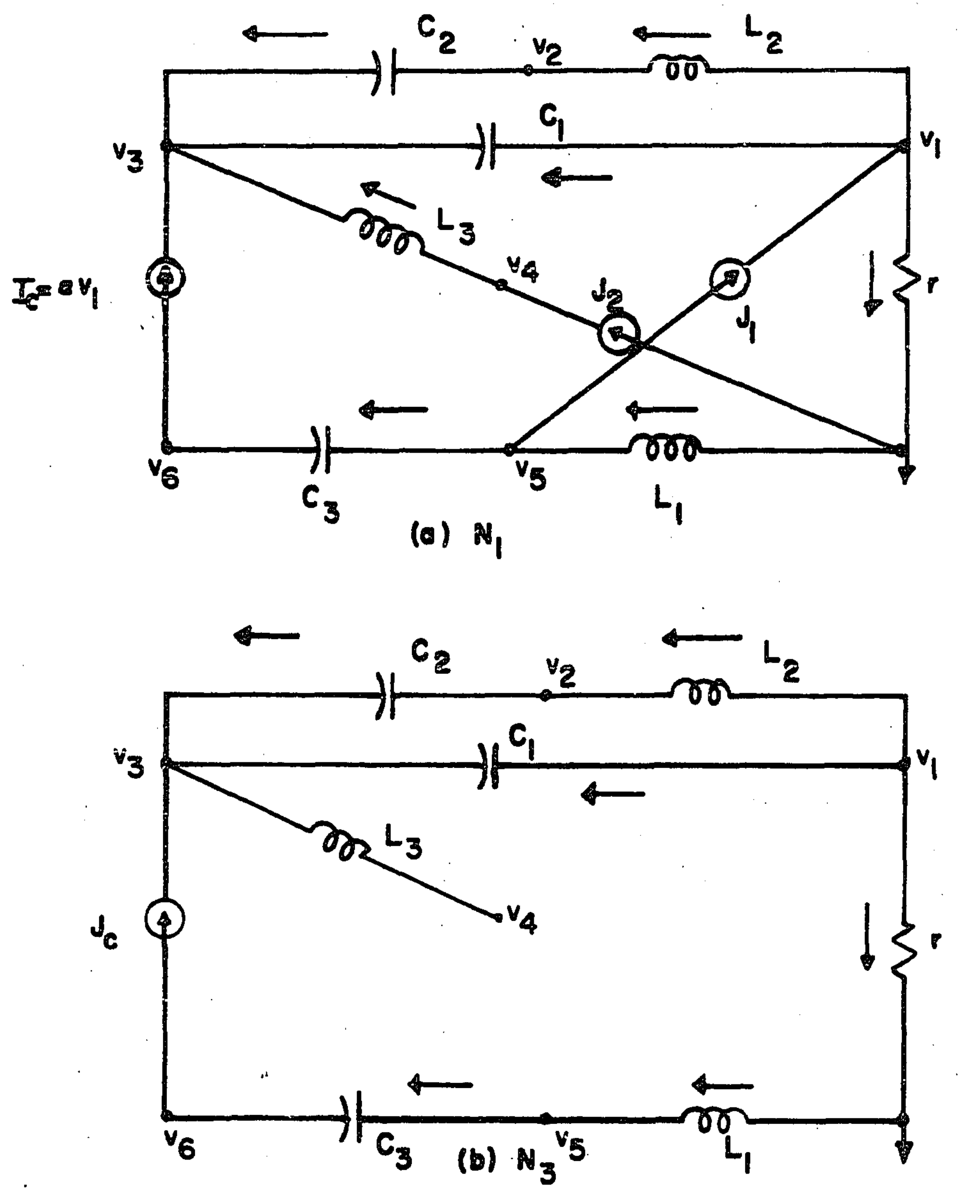


Figure 5. A network N_1 having a real transfer function, and the corresponding N_3

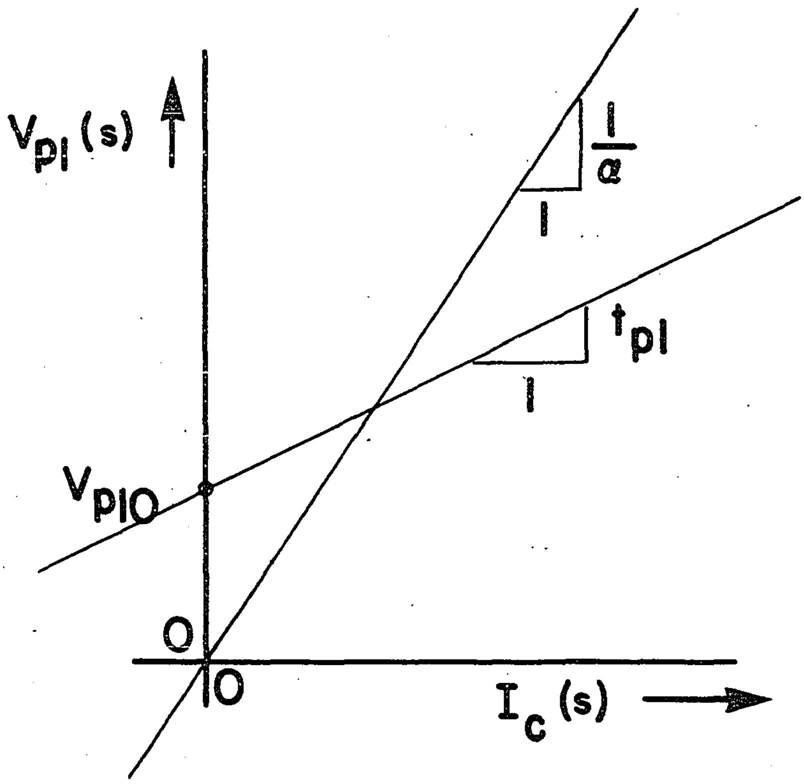


Figure 6. Straight line interpretation of condition 3

N_3, such as t_{p1}, may be found without evaluating determinants by means of topological formulas. These formulas also help in visualizing and proving certain properties of controlled-source networks. This will be demonstrated in the following discussion.

To use the topological formulas developed by Percival (12) directly, the fixed sources of N_3, excluding the replacement source, are properly removed. The result is a passive network driven by a single fixed source which is described by the same network functions as N_3. Removing the replacement source then leaves a network of passive elements having a pair of input vertices, a and b, to which the terminals of the replacement source were formerly connected, and a pair of output vertices, c and d, to which the controlling element is incident. The linear graph constructed such that there is a one-to-one correspondence between the graph edges and elements and between the graph vertices and the nodes of this network, is the graph to which Percival's topological formulas apply.

Condition 3 of Theorem 11 may be written in terms of the graph described above as

$$\frac{1}{\alpha} \neq \frac{\Sigma L_{ab,cd}}{\Sigma T} \tag{19}$$

or

$$\frac{1}{\alpha} \neq \frac{\Sigma T_{ac,bd} - \Sigma T_{ad,bc}}{\Sigma T}, \tag{20}$$

where the expressions on the right of Equations 19 and 20 are equivalent topological formulas for the transfer function t_{p1}.

Since α is a real number by Postulate P', $t_{p1} = 0$ is a sufficient condition for unique solvability of networks which satisfy conditions 1 and 2 of Theorem 11. Equation 20 suggests that there are two ways in which the transfer function may be zero; either both of the 2-tree sums of the numerator may be zero, or the first sum may equal the second. A condition sufficient to insure that both terms are zero is given by Theorem 12.

> Theorem 12. If a transformerless network N_1, satisfying conditions 1 and 2 of Theorem 11, is such that the terminal vertices of the replacement source are contained in a subgraph G of the graph G_3 of N_3, the controlling element is contained in the complement, G^*, of G, and if G shares exactly one vertex with G^*, then the network equations have a unique solution for all finite values of .

Proof. Let p be the single vertex common to G and G^* as illustrated in Figure 7. Suppose $\Sigma T_{ac,bd}$ is not zero. Then, corresponding to any term t of $\Sigma T_{ac,bd}$ is a pair of unconnected sub-trees, t_1 and t_2, of G_3, together containing all of the vertices of G_3, and such that t_1 contains vertices a and c and t_2 contains vertices b and d. But every path from a to c in G_3 contains p, so t_1 must contain p. Also, every path from b to d in G_3 contains p, so t_2 contains p. Therefore, t_1 and t_2 both contain p, which is impossible, for t_1 and t_2 are unconnected. Therefore, there is no such term t, and

$$\Sigma\, T_{ac,bd} = 0.$$

A similar argument proves that $\Sigma\, T_{ad,bc} = 0$. Therefore the right side of Equation 20 is zero, and the inequality holds for all finite α. This completes the proof.

The other way in which t_{pl} may be zero is illustrated by the graph shown in Figure 8. For this graph

$$\Sigma\, T_{ac,bd} = y_1 y_2 y_3 y_5$$

and

$$\Sigma\, T_{ad,bc} = y_1 y_2 y_4 y_6.$$

When the bridge is balanced, $y_4 y_6 = y_3 y_5$, and the numerator terms on the right side of Equation 20 cancel, making $t_{pl} = 0$.

For transformerless resistive networks, it is possible, using topological formulas, to derive a range for α in which condition 3 is always satisfied. If every passive element of N_1 is a resistor, then N_3 is a resistive network driven by the replacement source J_c, and y_{pl} of Figure 4 is a positive real number. Then, if $i_{ab,cd}$ denotes the current which flows through y_{pl} from d to c, according to Percival (12),

$$\frac{i_{ab,cd}}{J_c} = \frac{\Sigma\, K_{ab,cd}}{\Sigma\, T} = \frac{\Sigma\, K_{ab,cd}^{\;+} - \Sigma\, K_{ab,cd}^{\;-}}{\Sigma\, T}, \tag{21}$$

where $\Sigma\, K_{ab,cd}^{\;+}$ denoted the sum of tree products of all trees in $\Sigma\, T$ which contain a path from b to a through y_{pl} with paths bdca, and $K_{ab,cd}^{\;-}$ denotes the sum of tree products of those trees of $\Sigma\, T$ which include y_{pl}

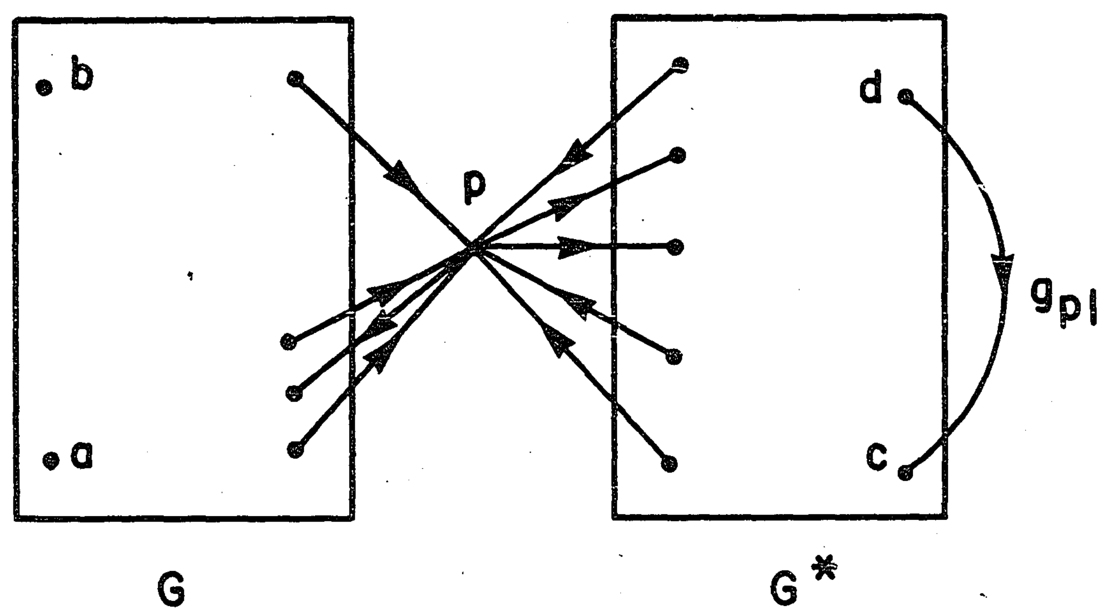


Figure 7. Graph illustrating Theorem 12

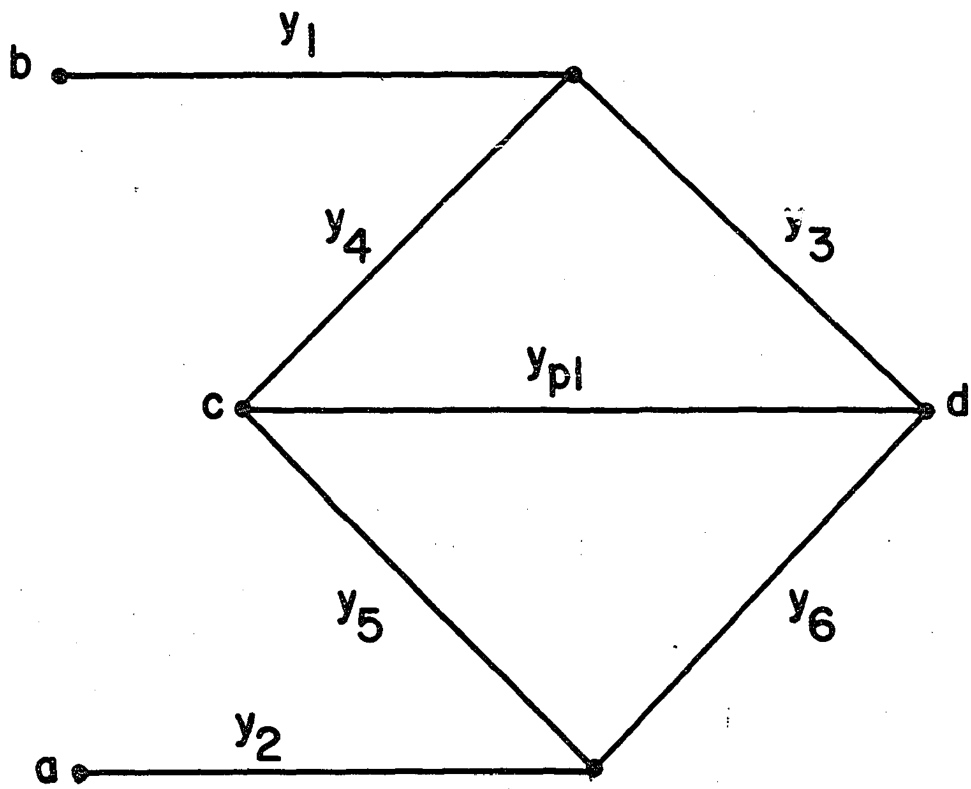


Figure 8. Graph illustrating term cancellation in an L-linkage

and have paths bcda. Since each of the terms in ΣT is positive, it follows that

$$0 \leq \Sigma K_{ab,cd}{}^{+} \leq \Sigma T$$

and

$$0 \leq \Sigma K_{ab,cd}{}^{-} \leq \Sigma T.$$

Then, using Equation 21, it may be concluded that

$$-1 \leq \frac{\Sigma K_{ab,cd}}{\Sigma T} \leq 1, \tag{22}$$

which is a constraint on the range of the transfer function of Equation 21. Of interest in the I-V_y case is the transfer function

$$t_{pl} = \frac{V_{ab,cd}}{J_c} = \frac{\Sigma L_{ab,cd}}{\Sigma T} = \frac{1}{y_{pl}} \quad \frac{\Sigma K_{ab,cd}}{\Sigma T}, \tag{23}$$

where $V_{ab,cd}$ is the element voltage of y_{pl}. Combining Equations 22 and 23 gives

$$-\frac{1}{y_{pl}} \leq t_{pl} \leq \frac{1}{y_{pl}}. \tag{24}$$

Then, for the I-V_y network described above to have no unique solution, it is necessary that $1/\alpha$ be in the range

$$-\frac{1}{y_{pl}} \leq \frac{1}{\alpha} \leq \frac{1}{y_{pl}}. \tag{25}$$

But, since   is positive by Postulate P',

$$0 < \frac{1}{\alpha} \leq \frac{1}{y_{p1}} . \tag{26}$$

Or, more simply, if

$$0 < \alpha < y_{p1} \tag{27}$$

the network equations always have a unique solution.

For transformerless single-source networks, it is possible to derive a useful topological formula for the system determinant. The derivation of this formula will close this discussion of single-source I-V_y networks.

Consider Equation 11, which describes a connected I-V_y network N_1. If N_1 contains no transformers, the submatrix Z_{p2} which appears in Equation 11 is diagonal, and the element-impedance matrix,

$$\begin{bmatrix} z_{p1} & 0 \\ 0 & Z_{p2} \end{bmatrix} ,$$

has an inverse,

$$\begin{bmatrix} y_{p1} & 0 \\ 0 & Y_{p2} \end{bmatrix} ,$$

which is the element-admittance matrix for the passive elements of N_1. By suitable row multiplications, it may be shown that the equation

$$\left[\begin{array}{c|ccccccc} -1/\alpha & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline A_c & A_{p1} & A_{p2} & A_E & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_c & B_{p1} & B_{p2} & B_J \\ 0 & -1 & 0 & 0 & 0 & y_{p1} & 0 & 0 \\ 0 & 0 & -U & 0 & 0 & 0 & Y_{p2} & 0 \end{array}\right] \left[\begin{array}{c} I_c \\ \hline I_{p1} \\ I_{p2} \\ I_E \\ V_c \\ V_{p1} \\ V_{p2} \\ V_J \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline -A_J J \\ -B_E E \\ I_{n1}' \\ I_{n2}' \end{array}\right], \tag{28}$$

where I_{n1}' and I_{n2}' represent the initial conditions obtained when admittance equations are used, is equivalent to Equation 11, and therefore describes N_1. Equation 28, rewritten in the concise form

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & M' \end{bmatrix} \begin{bmatrix} I_c \\ N \end{bmatrix} = \begin{bmatrix} 0 \\ D' \end{bmatrix}, \tag{29}$$

may be changed by the transformation

$$T_1' = \begin{bmatrix} 1 & 0 \\ 0 & (M')^{-1} \end{bmatrix} \tag{30}$$

into the equivalent system given by Equation 17 which, except for a single row operation, is the same as Equation 18. The transformation matrix which changes Equation 17 into Equation 18 has a determinant of one. Therefore, applying the rule for the determinant of the product of two matrices to the transformation of the coefficient matrices gives

$$(t_{p1} - \frac{1}{\alpha}) = \det\,(M')^{-1} \det \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & M' \end{bmatrix}. \tag{31}$$

But, since

$$\det (M')^{-1} = \frac{1}{\det M'} \ , \tag{32}$$

it follows that

$$\det \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & M' \end{bmatrix} = (\det M')(t_{p1} - 1/\alpha\ ) \ , \tag{33}$$

where the expression on the left is the determinant of the coefficient matrix of Equation 28. It was previously shown that $(t_{p1} - 1/\alpha\ )$ may be expressed by topological formulas in terms of trees and 2-trees of the linear graph of N_3. In the appendix it is proved that

$$\det (M') = \pm k' \Sigma T. \tag{34}$$

Also, Percival (12) states that

$$t_{p1} = \frac{\Sigma L_{ab,cd}}{\Sigma T} . \tag{35}$$

Substituting Equations 34 and 35 into Equation 33 gives

$$\det \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & M' \end{bmatrix} = \pm\ k' \ (\Sigma L_{ab,cd} - \frac{\Sigma T}{\alpha}) \tag{36}$$

where k' is an integer. Therefore, if network N_1 contains a single I-V_y source, has no transformers, and satisfies conditions 1 and 2 of Theorem 11, the system determinant may be expressed in terms of the simpler network N_3 by means of Equation 36.

The necessary and sufficient conditions for unique solvability of connected networks which satisfy the postulates and contain one controlled source of the $I\text{-}I_z$, the $V\text{-}I_z$, or the $V\text{-}V_y$ type are similar to the conditions of Theorem 11. Conditions 1 and 2 are common to all of these networks. For each network, there is a third condition which states that the reciprocal of the transmittance constant may not equal that transfer function of network N_3 which relates the controlling variable to the output of the replacement source. Each proof begins with an equation similar to Equation 11. Conditions 1 and 2 are proved necessary, as in Theorem 11, by using those columns of the coefficient matrix which contain only submatrices of A or B. In each case, a non-singular submatrix of the coefficient matrix is found, and used to construct a transformation matrix similar to that of Equation 15, leading to the third necessary condition. The sufficiency proofs follow the pattern of the proof of Theorem 11.

<u>Admittance-controlled n-source networks</u>

The equation for a connected network N_1 containing n $I\text{-}I_z$ sources and satisfying Kirchhoff's laws and Postulate P' may be written in the form

$$\left[\begin{array}{c|ccccccc} -\mathcal{L}^{-1} & U & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline A_c & A_{p1} & A_{p2} & A_E & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_c & B_{p1} & B_{p2} & B_J \\ 0 & Z_{p1} & 0 & 0 & 0 & -U & 0 & 0 \\ 0 & 0 & Z_{p2} & 0 & 0 & 0 & -U & 0 \end{array}\right] \left[\begin{array}{c} I_c \\ \hline I_{p1} \\ I_{p2} \\ I_E \\ V_c \\ V_{p1} \\ V_{p2} \\ V_J \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline -A_J J \\ -B_E E \\ I_{n1} \\ I_{n2} \end{array}\right], \qquad (36)$$

where:

$\mathcal{A}^{-1}$ is a diagonal matrix of order n with the reciprocals of the transmittance constants occupying the diagonal positions. The other matrices are as defined for Equation 11, except that I_c, I_{pl}, V_c, V_{pl}, and the corresponding submatrices of the coefficient matrix have their orders modified to accomodate n controlled sources and n controlling elements.

In the theorem which follows, it is assumed that the controlled sources, their corresponding controlling elements, and their transmittance constants are numbered consecutively from 1 to n.

Theorem 13. Equation 37 has a unique solution if and only if:

1. N_1 contains no cut-set consisting only of elements belonging to S_a.
2. N_1 contains no circuit consisting only of elements belonging to S_b.
3. $\left| F_{pl} - \mathcal{A}^{-1} \right| \neq 0$,

where F_{pl} is an nxn matrix such that element f_{ij} of F_{pl} is the transfer function of N_3 which relates the current of controlling element i to the current of replacement source j when all other replacement sources have been properly removed from N_3.

Proof. If the coefficient matrix of Equation 37 is non-singular, the columns including B_c and B_J must be linearly independent. But these columns correspond to the elements of S_a. Therefore, condition 1 of the theorem is necessary. Also, the columns containing A_E must be linearly independent, which establishes that condition 2 is necessary.

Conditions 1 and 2 guarantee that N_2 satisfies Theorem 10, and therefore that the network equations for N_2 have a unique solution.

Equation 37 may be rewritten in the form

$$\left[\begin{array}{c:c} m_{11} & m_{12} \\ \hdashline m_{21} & M \end{array}\right] \left[\begin{array}{c} I_c \\ \hdashline N \end{array}\right] = \left[\begin{array}{c} 0 \\ \hdashline D \end{array}\right], \tag{38}$$

where the dashed lines define the submatrices of Equation 38 in terms of the submatrices of Equation 37.

Denoting by J_c the n-rowed vector of replacement source currents N_2, the equation for N_2 may be written in the form

$$MN = D - m_{21}\, J_c \quad , \tag{39}$$

which has the unique solution

$$N = M^{-1}D - M^{-1}m_{21}\, J_c . \tag{40}$$

Using the non-singular transformation matrix

$$T_1 = \begin{bmatrix} U & 0 \\ 0 & M^{-1} \end{bmatrix},$$

where U is an identity matrix of order n, Equation 38 may be transformed by premultiplication into the equivalent equation

$$\begin{bmatrix} m_{11} & m_{21} \\ M^{-1}m_{21} & U \end{bmatrix} \begin{bmatrix} I_c \\ N \end{bmatrix} = \begin{bmatrix} 0 \\ M^{-1}D \end{bmatrix} . \tag{41}$$

The term $M^{-1} D$ on the right of Equation 41 is identified from Equation 40 as the solution vector for N_2 when every replacement source has a time function of zero. However this is, by definition of N_2, the solution vector for N_1 when all n controlled sources are replaced by current sources having time functions of zero. This is equivalent to stating that $M^{-1} D$ is the solution vector which would be obtained for N_1 if all of the controlled sources were properly removed.

In Equation 40, $-M^{-1} m_{21} J_c$ is recognized as the solution vector of N_2 for the special case in which $D = 0$. All sources in N_2 except the replacement sources then have time functions of zero and the initial conditions are zero. Therefore $-M^{-1} m_{21} J_c$ is the solution vector for N_3. Each row of this vector corresponds to one of the variables of N_3 and consists of a sum of n terms, one arising from each element of J_c, or each replacement source. This is simply a representation of each variable as a superposition of n responses to the replacement sources taken one at a time. Then the element in row i and column j of $-M^{-1} m_{21}$ is the transfer function which relates variable i of N_3 to replacement source j when every other replacement source has an output current specified to be zero, or has been properly removed. $M^{-1} m_{21}$ in Equation 41 is thus the matrix having the negatives of these transfer functions for elements.

It follows from the discussion above that Equation 41 may be written in the form

$$
\left[\begin{array}{c:ccccccc}
-\mathcal{A}^{-1} & U & 0 & 0 & 0 & 0 & 0 & 0 \\
\hdashline
-F_{p1} & U & 0 & 0 & 0 & 0 & 0 & 0 \\
-F_{p2} & 0 & U & 0 & 0 & 0 & 0 & 0 \\
-F_{E} & 0 & 0 & U & 0 & 0 & 0 & 0 \\
-T_{c} & 0 & 0 & 0 & U & 0 & 0 & 0 \\
-T_{p1} & 0 & 0 & 0 & 0 & U & 0 & 0 \\
-T_{p2} & 0 & 0 & 0 & 0 & 0 & U & 0 \\
-T_{J} & 0 & 0 & 0 & 0 & 0 & 0 & U
\end{array}\right]
\left[\begin{array}{c}
I_c \\ \hdashline I_{p1} \\ I_{p2} \\ I_E \\ V_c \\ V_{p1} \\ V_{p2} \\ V_J
\end{array}\right]
=
\left[\begin{array}{c}
0 \\ \hdashline I_{p10} \\ I_{p20} \\ I_{E0} \\ V_{c0} \\ V_{p10} \\ V_{p20} \\ V_{J0}
\end{array}\right], \qquad (42)
$$

where the F's are matrices of current/current transfer functions, the T's are matrices of voltage/current transfer functions, and the terms on the right are matrices of values the variables of N_1 assume when all controlled sources are removed from N_1. Both $\mathcal{A}^{-1}$ and $-F_{p1}$ are nxn matrices. By subtracting row n + i from row i, for i = 1, 2, ..., n in Equation 42, it may be shown that Equation 42 has a unique solution only if condition 3 of the theorem is satisfied. This completes the necessity portion of the theorem.

The sufficiency proof follows the pattern of the proof of Theorem 11 and is not presented here.

Conditions 1 and 2 of Theorem 13 constitute a set of constraints imposed upon the topological locations of the sources of N_1. Equation 42 shows that condition 3 of the theorem requires that the constraints imposed by the transmittance equations of the controlled sources be independent of another set of constraints imposed upon the same variables by the network in which the controlled sources are imbedded.

It is desirable to pursue the study further, searching for some network properties which imply and are implied by condition 3. Toward this end, the possibility of interpreting condition 3 in terms of vector spaces was explored without success. Another approach involves writing the determinant of condition 3 in an expanded form and attempting to interpret the coefficients in terms of network properties. For example, if N_1 contains three controlled sources, the determinant of condition 3 is given by

$$\left| F_{p1} - \mathcal{A}^{-1} \right| = \begin{vmatrix} f_{11} - 1/\alpha_1 & f_{12} & f_{13} \\ f_{21} & f_{22} - 1/\alpha_2 & f_{23} \\ f_{31} & f_{32} & f_{33} - 1/\alpha_3 \end{vmatrix} , \tag{43}$$

which is equal to

$$\begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} - \frac{1}{\alpha_1} \begin{vmatrix} f_{22} & f_{23} \\ f_{32} & f_{33} \end{vmatrix} - \frac{1}{\alpha_2} \begin{vmatrix} f_{11} & f_{13} \\ f_{31} & f_{33} \end{vmatrix} - \frac{1}{\alpha_3} \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} + \frac{f_{33}}{\alpha_1 \alpha_2}$$

$$+ \frac{f_{22}}{\alpha_1 \alpha_3} + \frac{f_{11}}{\alpha_2 \alpha_3} - \frac{1}{\alpha_1 \alpha_2 \alpha_3} . \tag{44}$$

Expression 44 lead to a number of new ideas which are discussed later in this chapter and in the following chapters. A special case in which each of the determinants in Equation 44 may be evaluated by topological formulas is discussed in detail in the section on new problems.

It is possible to derive theorems comparable to Theorem 13 for n-source networks of the I-V_y, V-I_z, or V-V_y types without introducing any new concepts. In each case conditions 1 and 2 apply exactly as stated in Theorem 13. Condition 3, as expected, differs in each case only in the nature of the transfer functions which constitute the elements of an nxn matrix similar to F_{p1}. These theorems are not stated and proved individually. Instead, the equations which describe the networks N_1 are first presented to show how conditions 1 and 2 arise, and how the coefficient matrix of the equation for N_2 appears as a submatrix of the coefficient matrix for N_1. This information is then used to prove a theorem for general admittance-controlled networks, which includes all four n-source networks as special cases.

The matrix equation for the n-source I-V_y case is

$$\left[\begin{array}{c|ccccccc} -\mathcal{A}^{-1} & 0 & 0 & 0 & 0 & U & 0 & 0 \\ \hline A_c & A_{p1} & A_{p2} & A_E & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_c & B_{p1} & B_{p2} & B_J \\ 0 & Z_{p1} & 0 & 0 & 0 & -U & 0 & 0 \\ 0 & 0 & Z_{p2} & 0 & 0 & 0 & -U & 0 \end{array}\right] \left[\begin{array}{c} I_c \\ \hline I_{p1} \\ I_{p2} \\ I_E \\ V_c \\ V_{p1} \\ V_{p2} \\ V_J \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline -A_J J \\ -B_E E \\ I_{n1} \\ I_{n2} \end{array}\right] \tag{45}$$

where S_a corresponds to the columns including B_c and B_J, and S_b to the columns including A_E. The submatrix at the lower right of the coefficient matrix, set off by dashed lines, is the coefficient matrix for the network N_2.

For the V-I_z case, the appropriate matrix equation is

$$\left[\begin{array}{c|ccccccc} -\mathcal{A}^{-1} & 0 & U & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & A_c & A_{p1} & A_{p2} & A_E & 0 & 0 & 0 \\ B_c & 0 & 0 & 0 & 0 & B_{p1} & B_{p2} & B_J \\ 0 & 0 & Z_{p1} & 0 & 0 & -U & 0 & 0 \\ 0 & 0 & 0 & Z_{p2} & 0 & 0 & -U & 0 \end{array}\right] \left[\begin{array}{c} V_c \\ \hline I_c \\ I_{p1} \\ I_{p2} \\ I_E \\ V_{p1} \\ V_{p2} \\ V_J \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline -A_J J \\ -B_E E \\ I_{n1} \\ I_{n2} \end{array}\right] . \qquad (46)$$

Note that the columns containing B_J correspond to S_a for this network, and that the columns containing A_c and A_E correspond to S_b, since the controlled sources, denoted by subscript c, are voltage sources in this case. The submatrix occupying the lower right corner of the coefficient matrix is the coefficient matrix for the equation which describe N_2.

Finally, the matrix equation for the n-source V-V_y case is

$$\left[\begin{array}{c|ccccccc} -\mathcal{A}^{-1} & 0 & 0 & 0 & 0 & U & 0 & 0 \\ \hline 0 & A_c & A_{p1} & A_{p2} & A_E & 0 & 0 & 0 \\ B_c & 0 & 0 & 0 & 0 & B_{p1} & B_{p2} & B_J \\ 0 & 0 & Z_{p1} & 0 & 0 & -U & 0 & 0 \\ 0 & 0 & 0 & Z_{p2} & 0 & 0 & -U & 0 \end{array}\right] \left[\begin{array}{c} V_c \\ \hline I_c \\ I_{p1} \\ I_{p2} \\ I_E \\ V_{p1} \\ V_{p2} \\ V_J \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline -A_J J \\ -B_E E \\ I_{n1} \\ I_{n2} \end{array}\right] . \qquad (47)$$

The columns containing B_J again correspond to S_a and the columns containing A_c and A_E again correspond to S_b. And, as usual, the submatrix at the lower right of the coefficient matrix is the coefficient matrix of the equation which describes N_2.

Networks containing admittance-controlled sources of all types

Using Equations 37, 45, 46, and 47, it is now possible to prove the following theorem.

Theorem 14. The equations for a connected network N_1 satisfying Kirchhoff's law and Postulate P', and containing

n_1 I-V_y sources,

n_2 I-I_z sources,

n_3 V-I_z sources,

n_4 V-V_y sources,

h fixed current sources,

k fixed voltage sources,

where $\sum_{i=1}^{4} n_i = n$, have a unique solution if and only if:

1. N_1 contains no cut-set consisting only of elements belonging to S_a.

2. N_2 contains no circuit consisting only of elements belonging to S_b.

3. $\left| K_{pl} - \mathcal{A}^{-1} \right| \neq 0$

where K_{pl} is an nxn matrix of transfer functions of N_3 such that element k_{ij} of K_{pl} is the ratio of the response of controlling variable i to the replacement source j when all other replacement sources have been properly removed from N_3, and $\mathcal{A}^{-1}$ is the nth order diagonal matrix having the reciprocals of the transmittance constants as diagonal terms.

Proof. The network equation for N_1 may be written in the form

$$\begin{bmatrix} -\mathcal{K}^{-1} & m_{12} \\ m_{21} & M \end{bmatrix} \begin{bmatrix} V_c \\ V \end{bmatrix} = \begin{bmatrix} 0 \\ D \end{bmatrix} \tag{48}$$

where:

V_c is the vector of controlled variables.

V is the vector of the remaining variables.

D is the vector of driving functions, each prefixed by the appropriate columns of the A or B matrix, and the initial conditions.

The first n equations are the transmittance constraints.

The remaining equations are the impedance and Kirchhoff's law equations.

With no loss in generality, the first n rows of V may be assumed to correspond to the controlling variables, with the row order such that m_{12} contains an nth order identity matrix at the extreme left and zeros elsewhere.

In each of the Equations 36, 45, 46, and 47, those columns of the coefficient matrix which correspond to the voltage variables of current sources, both fixed and controlled, contain only submatrices of the B matrix. Also, those columns which correspond to the current variables of voltage sources, both fixed and controlled, contain only submatrices of the A matrix. Inspection of the equations makes it evident that the same observations would be made if space permitted Equation 48 to be written in a form showing the detailed partitioning. This is because all of the identity and impedance matrices of the coefficient matrix must appear in the columns associated with the passive elements, leaving no nonzero entries for the source columns other than submatrices of A and B. Thus, the arguments used previously establish that conditions 1 and 2 are necessary and, therefore, that the network equation for N_2 has a unique solution.

Using V_c' for the vector of replacement sources, the equation for N_2 is

$$M\ V = D - m_{21}\ V_c', \tag{49}$$

and the unique solution to Equation 49 is given by

$$V = M^{-1}\ D - M^{-1} m_{21}\ V_c'. \tag{50}$$

The non-singular transformation

$$T_1 = \begin{bmatrix} U & 0 \\ 0 & M^{-1} \end{bmatrix}, \tag{51}$$

where U is the order n, may be used to transform Equation 48 into the equivalent equation

$$\begin{bmatrix} -\mathcal{A}^{-1} & m_{12} \\ M^{-1} m_{21} & U \end{bmatrix} \begin{bmatrix} V_c \\ V \end{bmatrix} = \begin{bmatrix} 0 \\ M^{-1} D \end{bmatrix}. \tag{52}$$

From Equation 50 and the structural similarity of N_1 and N_2, it follows that $M^{-1}D$ of Equation 52 is tha matrix of values for the variables of N_1 when the controlled sources are properly removed from N_1.

Equation 50 also indicates that $-M^{-1}m_{21}\ V_c'$ is a matrix of solutions to the equations of N_3, expressing each variable as a superposition of responses to the n replacement sources of V_c'. Then $-M^{-1}m_{21}$ of Equation 50 is a matrix in which the element in row i and column j is the transfer function which relates variable i of N_3 to replacement source j when every other replacement source has been properly removed. In particular, the first n rows of $-M^{-1}m_{21}$ contain the transfer functions which correspond to the controlling variables of N_3. This is the submatrix K_{p1} of condition 3 of the theorem.

Now, the nth order identity matrix at the extreme left of m_{12} in Equation 52 may be eliminated by subtracting row n + i from row i, for i = 1, 2, ..., n. The result is a coefficient matrix, M_c, having the form

$$M_c = \begin{bmatrix} (K_{p1} - \mathcal{A}^{-1}) & 0 & 0 \\ -K_{p1} & U & 0 \\ -K_o & 0 & U \end{bmatrix}, \qquad (53)$$

where

$$\left.\begin{matrix} -K_{p1} \\ -K_o \end{matrix}\right] = M^{-1}m_{21}.$$

The equivalence of Equations 48 and 52 and the non-singularity of the transformation relating the coefficient matrix of Equation 52 to M_c of Equation 53 establish that condition 3 is necessary.

Conditions 1 and 2 imply the non-singularity of the transformation matrix, T_1 of Equation 51, which in turn guarantees the equivalence of Equations 48 and 52. But the determinant of the coefficient matrix of Equation 52 is $\left| K_{p1} - \mathcal{A}^{-1} \right|$. Therefore conditions 1, 2, and 3 are sufficient, which completes the proof.

Using conditions 1 and 2 of Theorem 14, it is possible to deduce upper limits for the number of current and voltage sources permissible if the network equation is to have a unique solution. By Seshu and Reed (16), conditions 1 and 2 are equivalent to stating that there is a tree in N_1 such that the elements of S_a are chords for this tree and the elements of S_b are branches for this tree. But if N_1 contains a total of e elements and v vertices, every tree of N_1 contains v-1 branches and e-v+1 chords. If conditions 1 and 2 are satisfied, it then follows that

$$n_1 + n_2 + h \leq e - v + 1 \tag{54}$$

and

$$n_3 + n_4 + k \leq v - 1. \tag{55}$$

<u>Networks controlled by open or short-circuit variables</u>

When networks contain sources controlled by open or short-circuit variables, some new complications arise in deriving the solvability conditions. Although the proofs must be modified slightly, the solvability conditions for networks containing $V\text{-}V_o$ or $I\text{-}I_o$ sources are exactly those expected in light of the cases considered so far. However, the conditions that might be expected for the $V\text{-}I_o$ and $I\text{-}V_o$ cases are sufficient but not necessary.

The equation for a connected network N_1 containing n $V\text{-}V_o$ sources may be written in the form

$$\left[\begin{array}{c|cccccc} -\mathcal{L}^{-1} & 0 & 0 & 0 & U & 0 & 0 \\ \hline 0 & A_c & A_{p2} & A_E & 0 & 0 & 0 \\ B_c & 0 & 0 & 0 & B_j & B_{p2} & B_J \\ 0 & 0 & Z_{p2} & 0 & 0 & -U & 0 \end{array}\right] \left[\begin{array}{c} V_c \\ \hline I_c \\ I_{p2} \\ I_E \\ V_j \\ V_{p2} \\ V_J \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline -A_j I_j - A_J J \\ -B_E E \\ I \end{array}\right] \qquad (56)$$

where V_j is the matrix of the controlling current sources which have currents I_j specified to be zero, and I is the set of initial conditions for N_1.

Theorem 15. Equation 56 has a unique solution if and only if:

1. N_1 contains no cut-set consisting only of elements belonging to S_a.

2. N_1 contains no circuit consisting only of elements belonging to S_b.

3. $\left|K - \mathcal{L}^{-1}\right| \neq 0$

where element k_{ij} of K is the transfer function relating controlling variable i to replacement source j of N_3 when all other replacement sources are properly removed from N_3.

Proof. If the coefficient matrix of Equation 56 is non-singular, the v-1 rows containing $[A_c\ A_{p2}\ A_E]$ must be linearly independent. Then, by Theorem 3, the elements of N_1 corresponding to the columns of $[A_j\ A_J]$, (which are exactly the elements of S_a), must belong to a chord set of some tree T_1. But if any subset of the elements of S_a constitute a cut-set, then, according to Seshu and Reed (16), this subset contains at least one branch of every tree. But this is impossible if the elements of S_a

are in the chord set of T_1. Therefore, N_1 may contain no cut-set of elements belonging to S_a, and condition 1 is necessary.

For the coefficient matrix to be non-singular, it is also necessary that the columns containing $[A_c\ A_E]$ be linearly independent. This implies that N_1 may contain no circuit consisting only of elements of S_b, proving that condition 2 is necessary.

Condition 1 and 2, as usual, establish that the network equation for the corresponding N_2 has a unique solution. The remainder of the proof follows the pattern established in the previous proofs.

The equation describing a connected network N_1 which contains n I-I_o sources may be written in the form

$$\left[\begin{array}{c|cccccc} -\mathcal{A}^{-1} & U & 0 & 0 & 0 & 0 & 0 \\ \hline A_c & A_e & A_{p2} & A_E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_c & B_{p2} & B_J \\ 0 & 0 & Z_{p2} & 0 & 0 & -U & 0 \end{array}\right] \left[\begin{array}{c} I_c \\ \hline I_e \\ I_{p2} \\ I_E \\ V_c \\ V_{p2} \\ V_J \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline -A_J J \\ -B_e V_e - B_E E \\ I \end{array}\right] \qquad (57)$$

where I_e is the matrix of currents of the controlling voltage sources for which $V_e = 0$.

Theorem 16. Equation 57 has a unique solution if and only if:

1. N_1 contains no cut-set consisting only of elements belonging to S_a.
2. N_1 contains no circuit consisting only of elements belonging to S_b.
3. $\left| F - \mathcal{A}^{-1} \right| \neq 0$

where element f_{ij} of F is the transfer function relating controlling variable i of N_3 to replacement source j when all other replacement sources are properly removed.

Proof. If Equation 57 has a unique solution, the columns of the coefficient matrix containing $[B_c \; B_J]$ must be linearly independent. This implies that no subset of the elements corresponding to these columns may constitute a cut-set. Therefore, N_1 contains no cut-set of elements belonging to S_a.

Similarly, those rows of the coefficient matrix containing $[B_c \; B_{p2} \; B_J]$ must be linearly independent. Therefore, there is some tree T which contains the elements corresponding to the columns of B_e and B_E, which are the elements of S_b. But a subset of elements of a tree may contain no circuit. Therefore N_1 contains no circuit consisting entirely of elements belonging to S_b.

The remainder of the proof is omitted because of its similarity to previous proofs.

Next consider a connected network N_1 satisfying Postulate P' and containing n I-V_o sources. The usual three conditions are seen to be sufficient but not necessary to guarantee the unique solvability of the network equation for N_1.

The network equation may be written in the form

$$\left[\begin{array}{c|cccccc} -\mathcal{L}^{-1} & 0 & 0 & 0 & U & 0 & 0 \\ \hline A_c & A_{p2} & A_E & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_c & B_j & B_{p2} & B_J \\ 0 & Z_{p2} & 0 & 0 & 0 & -U & 0 \end{array}\right] \left[\begin{array}{c} I_c \\ \hline I_{p2} \\ I_E \\ V_c \\ V_j \\ V_{p2} \\ V_J \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline -A_j I_j - A_J J \\ -B_E E \\ I \end{array}\right] . \tag{58}$$

If N_1 contains no cut-set consisting entirely of elements from S_a and no circuit consisting entirely of elements from S_b, then N_2 satisfies the conditions of Theorem 10, and the network equation for N_2 has a unique solution. Therefore, a non-singular transformation matrix may be constructed to transform the coefficient matrix of Equation 58 into an equivalent matrix having a determinant

$$\left| K - \mathcal{L}^{-1} \right| ,$$

where K is the nxn matrix of transfer functions of N_3 which relate the controlling voltages to the replacement currents. Therefore, the usual three conditions are sufficient to guarantee the unique solvability of Equation 58.

It is convenient to think of set S_a as consisting of the union of three mutually exclusive subsets. If S_{a1} denotes the set of all fixed current sources which are not controlling elements, S_{a2} the set of all fixed current sources which are controlling elements, and S_{a3} the set of controlled current sources, then

$$S_a = S_{a1} \cup S_{a2} \cup S_{a3} \tag{59}$$

where U, in this context, denotes a union of sets. Set S_b contains only the fixed voltage sources.

If the coefficient matrix of Equation 58 is non-singular, the columns containing A_E must be independent. This implies that N_1 may contain no circuit consisting only of elements belonging to S_b. Also, if the coefficient matrix is non-singular, the rows containing $[A_c\ A_{p2}\ A_E]$ must be linearly independent, which implies that N_1 contains no cut-set consisting entirely of elements of $S_{a1} \cup S_{a2}$. Finally, for the coefficient matrix to be non-singular, it is necessary that the columns containing $[B_c\ B_J]$ be linearly independent. This implies that N_1 contains no cut-set consisting entirely of elements belonging to $S_{a1} \cup S_{a3}$. Therefore, if the equation for N_1 has a unique solution, it is necessary that N_1 contain no cut-set consisting entirely of elements belonging to S_{a1}, S_{a2}, S_{a3}, $S_{a1} \cup S_{a2}$, or $S_{a1} \cup S_{a3}$. Conspicuous by their absence are any obvious arguments forbidding a cut-set of elements from $S_{a2} \cup S_{a3}$ or from $S_{a1} \cup S_{a2} \cup S_{a3}$. Thus the usual location constraints do not appear to be necessary for unique solvability in this case.

Examining the submatrix at the lower right of the coefficient matrix of Equation 58 reveals an unanticipated difficulty. As usual, this submatrix is the coefficient matrix for the network equation of N_2. If N_1 is a uniquely solvable network which contains a cut-set of elements, belonging to $S_{a2} \cup S_{a3}$, but not to S_{a2} or S_{a3} alone, then there exists some linear combination of the corresponding columns, some in B_c and some in B_j, which sums to zero. This means that the submatrix is singular. Therefore, it is possible that the equation for network N_1 might have a unique solution even though the equation for N_2 does not. Since the coefficient matrix of the equation for N_2 may thus be singular for a solvable

N_1, a transformation matrix similar to those used in previous theorems may not be constructed to search for another necessary condition.

Figure 9 shows a network N_1 which illustrates the problem. The elements of S_{a2} are the sources connected to the rest of the network by dashed lines. Nodal analysis of N_1 shows that the system determinant is given by

$$\Delta = -y_a y_b y_c y_d \, (\alpha_1 + \alpha_2), \tag{60}$$

which is nonzero for positive α_1 and α_2. However, the elements of $S_{a2} \cup S_{a3}$ constitute a cut-set, which is sufficient to guarantee that the equation for N_2 has no unique solution.

A study of the equations for connected, n-source, $V\text{-}I_o$ networks shows that the usual three conditions are sufficient but not necessary for unique solvability in this case also. N_1 may include circuits containing both controlling voltage sources and controlled voltage sources and still have a uniquely solvable system equation. Since the arguments leading to these results closely parallel the arguments used in the $I\text{-}V_o$ case, they are not presented here.

The problem of finding a complete set of necessary and sufficient conditions for the $I\text{-}V_o$ and $V\text{-}I_o$ n-source cases has not been solved. However, an approach to the solvability problem using general topological formula methods is suggested in the new problems section. This approach is potentially general enough to apply to all connected networks which satisfy Postulate P', even the two just discussed above, since the method does not depend upon a transformation matrix.

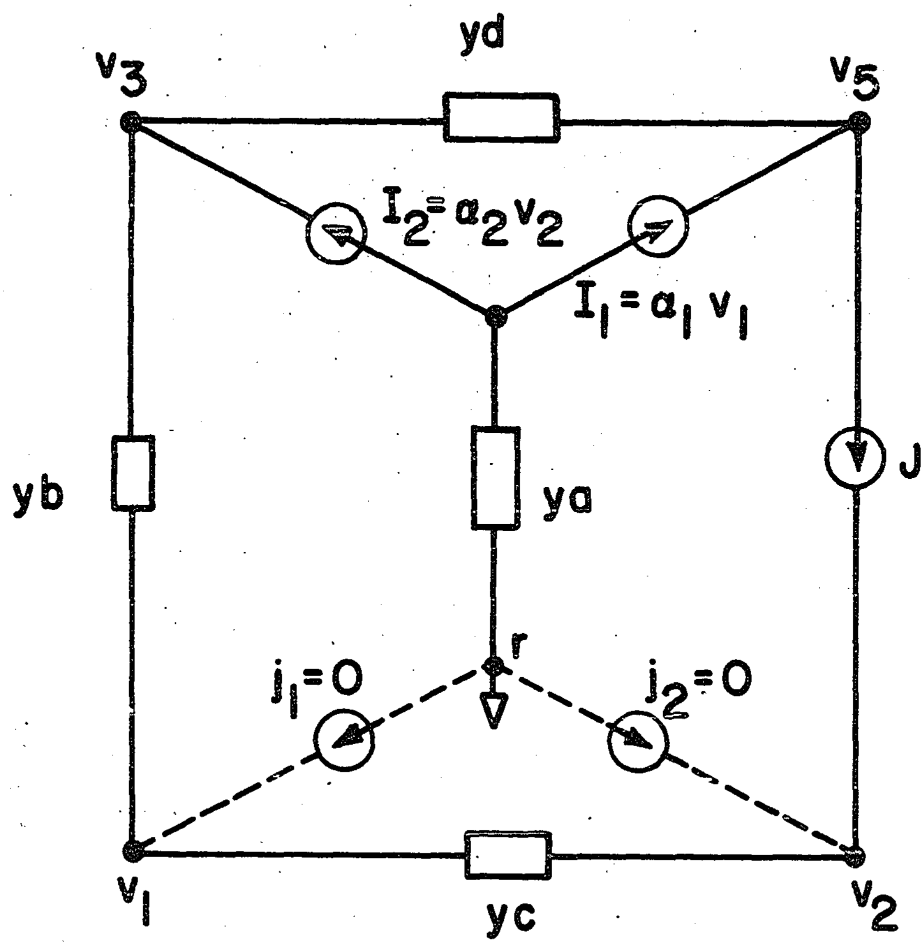


Figure 9. Network having a unique solution for N_1 but not for N_2

<u>Networks with transmittance constants which approach infinity</u>

Controlled-source networks in which the values of the transmittance constants are assumed to approach infinity constitute the final topic of this chapter. A network containing one $V\text{-}V_o$ source with a very high transmittance constant is used to represent a high-gain operational amplifier network. A topological formula derived here for this network is compared with a formula derived by Sinha, and the two approaches are further compared in an example. Finally, two theorems from the last chapter are used to derive some location constraints for the controlling elements of an n-source $I\text{-}I_z$ network having transmittance constants which approach infinity. Only if these constraints are satisfied will the network equation remain uniquely solvable in the limit.

Sinha (17) deduces from the work of Nathan (8,9), that the nodal determinant, Δ, for a passive transformerless network containing one "infinite gain" operational amplifier is given, except for a sign factor, by the topological formula

$$\Delta = \Sigma\, T_{bd,r}, \tag{61}$$

where r is the reference node of the operational amplifier, and b and d are its input and output nodes respectively.

Now consider a passive, transformerless network in which is imbedded a single $V\text{-}V_o$ source having a transmittance constant α, and such that the controlled source is incident to vertices a and b and the controlling element is incident to vertices c and d. Since the $V\text{-}V_o$ source draws no current at the input node, this source type may be used as a model for a high-gain operational amplifier by adding suitable restrictions.

Using the results of Theorem 15, an expression similar to that given by Equation 33 may be derived for the single-source $V\text{-}V_o$ case which expresses the determinant of the coefficient matrix Δ' in the form

$$\Delta' = (\det M')(k_{pl} - 1/\alpha), \tag{62}$$

where $(\det M')$ is the determinant of the coefficient matrix of the equation describing N_2, and k_{pl} is the transfer function relating the controlling voltage of N_3 to the output of the replacement source. For a transformerless connected network, Percival states that

$$k_{pl} = \frac{\Sigma\, L_{ab,cd}}{\Sigma\, T_{a,b}} = \frac{\Sigma\, T_{ac,bd} - \Sigma\, T_{ad,bc}}{\Sigma\, T_{a,b}}\,. \tag{63}$$

Since Sinha's operational amplifier is a 3-terminal device, vertices a and c of the $V\text{-}V_o$ source-controlling element pair will be assumed identical, and this common vertex will be denoted by r. Then Equation 63 becomes

$$k_{pl} = \frac{\Sigma\, T_{r,bd} - \Sigma\, T_{rd,br}}{\Sigma\, T_{r,b}}\,, \tag{64}$$

which reduces to

$$k_{pl} = \frac{\Sigma\, T_{r,bd}}{\Sigma\, T_{r,b}}\,, \tag{65}$$

since the sum of admittance products for a set of 2-trees required to contain a node r in both sub-trees is zero. Substituting Equation 65 into Equation 62 gives

$$\Delta' = (\det M') \left(\frac{\Sigma\, T_{r,bd}}{\Sigma\, T_{r,b}} - \frac{1}{\alpha} \right). \tag{66}$$

Arguments similar to those used in the appendix may be used to prove that

$$(\det M') = \pm\ k'\ \Sigma\, T_{r,b}\ , \tag{67}$$

where k' is an integer. Then Equation 66 becomes

$$\Delta' = \pm\, k' \left(\Sigma\, T_{r,bd} - \frac{\Sigma\, T_{r,b}}{\alpha} \right). \tag{68}$$

Finally, to represent Sinha's operational amplifier, the transmittance constant of the $V\text{-}V_o$ source must approach infinity. This gives

$$\lim_{\alpha \to \infty} \Delta' = \pm\, k'\ \Sigma\, T_{r,bd} \tag{69}$$

which, except for the factor k', agrees with Sinha's result. Since the factor k' is common to every term on the right side of Equation 69, it is of no consequence in a practical sense. k' is a result of using the circuit matrix B explicitly in the derivation instead of using only **A** and its transpose as in the nodal approach. If the general network equations which were used in the derivations were used to calculate the response of the operational amplifier network to some driving source, the factor k' would also appear in the numerator, and the two (k')'s would cancel. A good discussion of the factor k' is given by Cederbaum (3). For our purposes, it suffices to cite the result given by Okada (10), that k' equals one if the rows of B represent fundamental circuits or meshes of a planar graph. Since no restriction was previously placed on B, there is no reason why these special circuits may not be used in defining B, so that the result agrees exactly with Sinha's.

The network used in Sinha's example and shown in Figure 10a may be represented by the controlled-source network of Figure 10b. The nodal equation for this network is

$$\begin{bmatrix} g_1 & -g_1 & 0 \\ -(g_1 + g_3 + g_2 + s\ C_1) & g_1 & -g_3 + 1/\alpha\ g_2 \\ -g_2 & 0 & \frac{g_2}{\alpha} + (1/\alpha + 1)\ s\ C_2 \end{bmatrix} \begin{bmatrix} V_2(s) \\ V_3(s) \\ E(s) \end{bmatrix} = 0. \qquad (70)$$

If Δ' denotes the determinant of the coefficient matrix, evaluation of Δ' gives

$$-\Delta' = (g_1 g_3\ s\ C_2 + g_1 g_2\ s\ C_2 + g_1\ s^2\ C_1 C_2 + g_1 g_2 g_3) +$$

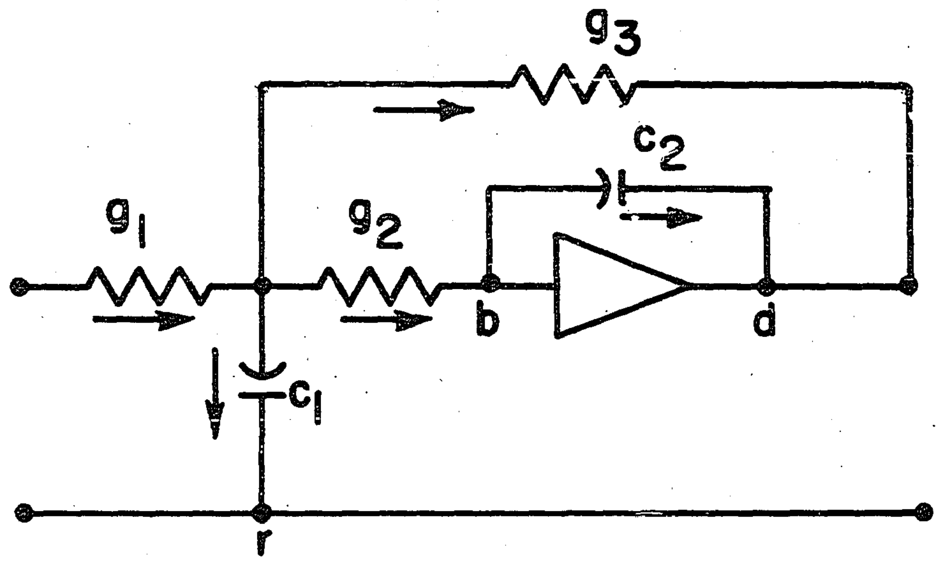
$$\frac{1}{\alpha}(g_1 g_2 g_3 + g_1 g_3\ s\ C_2 + g_1 g_2\ s\ C_2 + g_1 g_2\ s\ C_1 + g_1\ s^2\ C_1 C_2). \qquad (71)$$

Now

$$\lim_{\alpha \to \infty} \Delta' = -(g_1 g_2 g_3 + g_1 g_2\ s\ C_2 + g_1\ g_3\ s\ C_2 + g_1\ s^2\ C_1 C_2), \qquad (72)$$

which is the result given by Sinha.

It is easy to verify by direct calculations that the first term on the right side of Equation 71 and the coefficient of $1/\alpha$ are, respec-



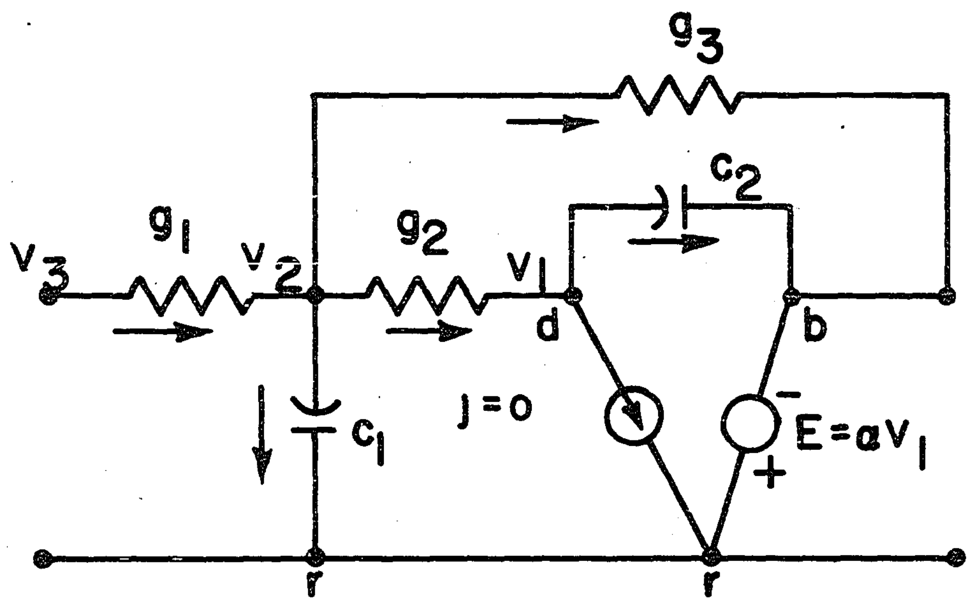


Figure 10. Two representations of an operational amplifier network

tively, the numerator and denominator of the transfer function

$$t_{pl} = \frac{V_1 (s)}{E (s)} \tag{73}$$

of the network of Figure 11a, as predicted by the theory.

Figure 11b shows the linear graph of Figure 11a which may be used to find the terms of Equation 71 by using topological formulas. Figure 11c shows the 2-trees of the graph which correspond to the coefficient of 1/ of Equation 71. Note that the first term on the right of Equation 71 corresponds to the set of all 2-trees of Figure 11c such that b and d are in one connected part and r is in the other, which was predicted by the theory.

The significance of the comparison with Sinha's result is clear. Nathan's approach, upon which Sinha's result is based, applies only to networks containing $V\text{-}V_o$ or $V\text{-}V_y$ sources for which the controlling and controlled elements share a common vertex. Sinha's result then describes a special case which may be found from the general theory presented here by (a) restricting the type of controlled source, (b) requiring a 3-terminal controlled source-controlling element configuration, (c) assuming $\alpha \rightarrow \infty$. In addition to presenting the result in a broad context, the general theory provides additional insight, since it identifies Sinha's result as the topological formula for the numerator of a particular transfer function of a related network. This identification of the formula was not revealed using the Nathan-Sinha approach.

Although Postulate P' requires that the transmittance constants be real numbers, the example discussed above stimulates interest in what

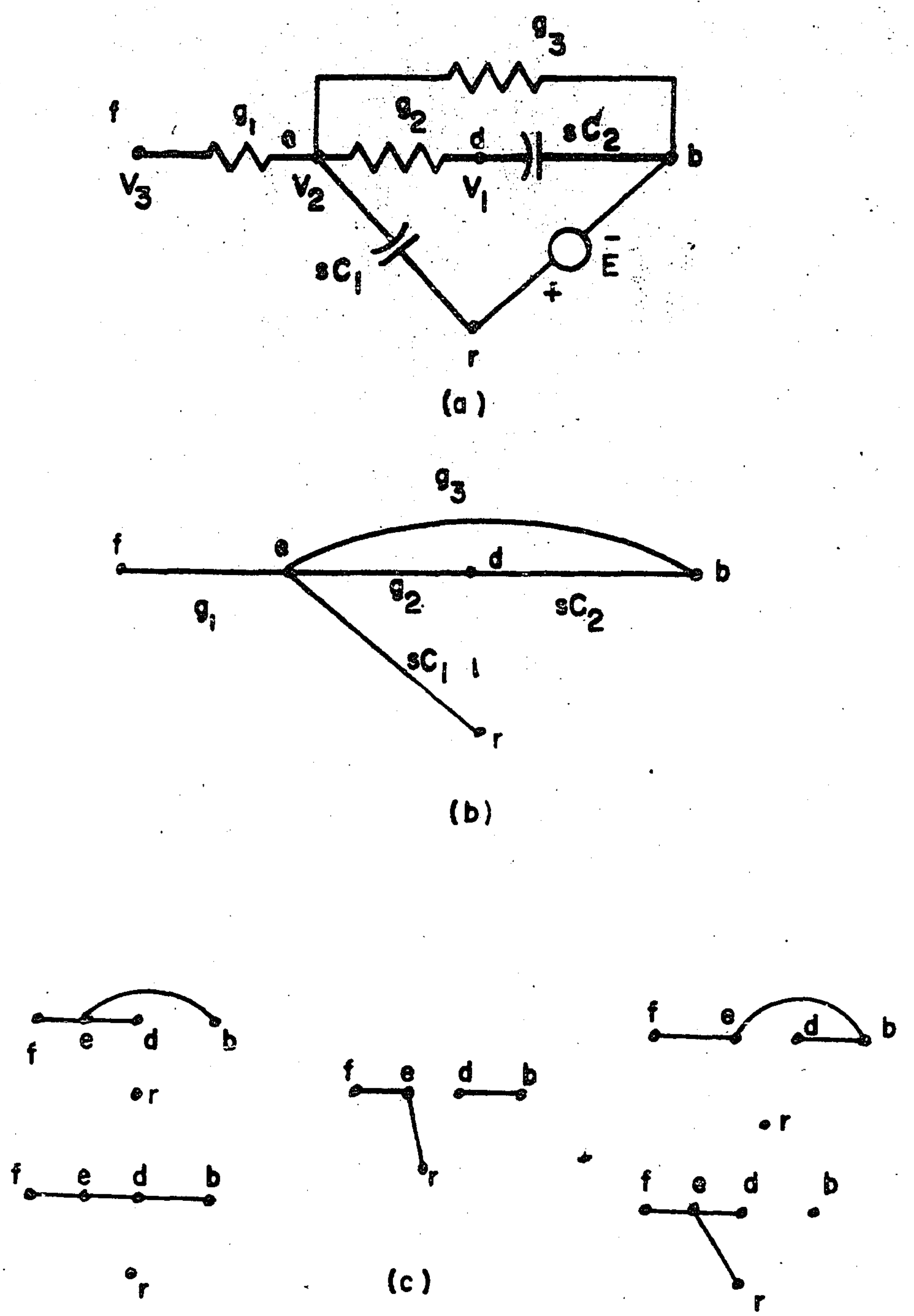


Figure 11. Network and graphs for interpreting Equation 72

happens to the solvability conditions in the limit as the transmittance constants approach infinity. The following two theorems give conditions in which the system determinant for a transformerless network satisfying the conditions of Theorem 13 become identically zero in the limit as some of the transmittance constants approach infinity.

Theorem 17. Let N_1 be a connected, transformerless network containing n consecutively numbered I-I_z sources and satisfying the three conditions of Theorem 13. If the s, $1 \leq s \leq n$, controlling elements which correspond to the controlled sources u_1, u_2, ..., u_s constitute a seg of N_1, and if U_s denotes a subset of the first n positive integers which contains u_1, u_2, ..., u_s, then the determinant of condition 3 of Theorem 13 approaches zero in the limit as the values of the transmittance constants corresponding to U_s approach infinity.

Proof. As the transmittance constants corresponding to U_s approach infinity, the corresponding terms of the diagonal matrix $\mathcal{A}^{-1}$ approach zero, leaving some of the rows of $[F_{pl} - \mathcal{A}^{-1}]$ containing no term from $\mathcal{A}^{-1}$. In particular, rows u_1, u_2, ..., u_s contain no term from $\mathcal{A}^{-1}$. Since N_1 contains no transformers, each term f_{ij} of F_{pl} may be written in terms of topological formulas as a ratio of a K-linkage to a tree sum. Furthermore, the denominators of all terms are identical, since each is the tree sum formed from N_3 after all sources have been properly removed. But then it follows from Theorem 9 that

$$\sum_{k=1}^{s} f_{u_k j} = 0 \quad \text{for all } j. \tag{74}$$

This proves the theorem, since the sum of s rows of

$$\lim_{\substack{\alpha_i \to \infty \\ \alpha_i \ \eta_i \ni U_s}} \left| F_{pl} \quad -\mathcal{A}^{-1} \right|$$

(where $\alpha_i \ \eta_i \ni U_s$ means the set of all transmittance constants α_i such that i is included in U_s) is zero.

Theorem 17 is actually a rather complicated method of expressing a simple consequence of Kirchhoff's current law, as illustrated by the following example.

If $n = 4$, $s = 2$, $u_1 = 1$, and $u_2 = 3$, network N_2 may be represented as shown in Figure 12. In this figure, the four replacement sources are explicitly shown, as are the controlling elements of sources 1 and 3 which constitute the seg. The other two controlling elements are not visible.

The determinant of condition 3 in this case may be written in the form

$$\Delta = \begin{vmatrix} f_{11} - \frac{1}{\alpha_1} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} - \frac{1}{\alpha_2} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} - \frac{1}{\alpha_3} & f_{34} \\ f_{41} & f_{42} & f_{43} & f_{44} - \frac{1}{\alpha_4} \end{vmatrix} . \tag{75}$$

If $U_s = (1, 2, 3)$, then

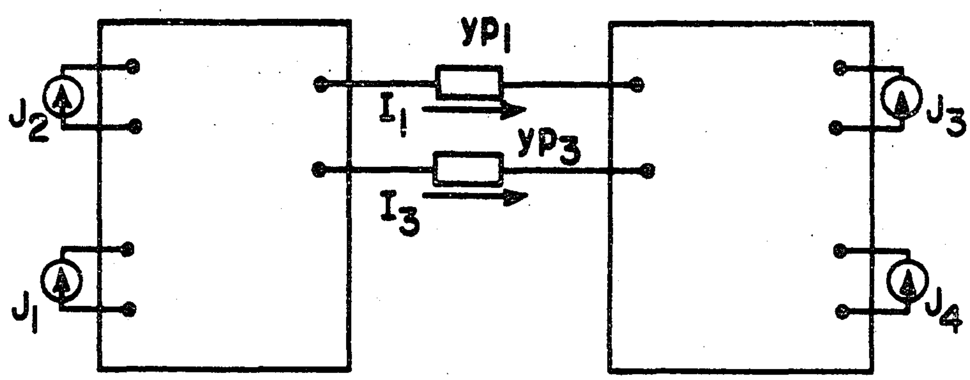


Figure 12. Network containing a seg of controlling elements

$$\lim_{\alpha_1, \alpha_2, \alpha_3 \to \infty} \Delta = \begin{vmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{41} & f_{42} & f_{43} & f_{44} - \frac{1}{\alpha_4} \end{vmatrix} . \tag{76}$$

The theorem states that

$$\begin{aligned} f_{11} &+ f_{31} = 0, \\ f_{12} &+ f_{32} = 0, \\ f_{13} &+ f_{33} = 0, \\ f_{14} &+ f_{34} = 0, \end{aligned} \tag{77}$$

which guarantees that the determinant of Equation 76 is zero, since two of its rows sum to zero.

Physically, the quantity $(f_{11} + f_{31})$ of Equation 77 is the sum of the values of I_1 and I_3 of Figure 12 when

$$J_1 \neq 0, \quad J_2 = J_3 = J_4 = 0. \tag{78}$$

But this sum is known to be zero from Kirchhoff's current law (15). The same reasoning also justifies the other three equalities of Equation 77.

An intuitive feel for the theorem may be gained by noting that each controlled variable, I_{ci}, is related to its controlling variable, I_{pi}, by an equation having the form

$$I_{ci} = \alpha_i \, I_{pi}. \tag{79}$$

In the limit, as $\alpha_i \longrightarrow \infty$, the controlling variable, I_{pi}, approaches zero and I_{ci} remains finite, reminiscent of the familiar assumption of zero grid voltage used in analyzing high-gain operational amplifier networks. Now Kirchhoff's current law requires that the sum of the currents of the elements of a seg be zero. But as the transmittance constants corresponding to the elements of the seg approach infinity, each current and hence their sum is required to approach zero by the foregoing argument. In essence then, the transmittance constraints and Kirchhoff's current law, which are expressed by separate equations in the general network equation, become linearly dependent, and the system determinant approaches zero.

As might be expected, Kirchhoff's voltage law leads to a second theorem similar in its form and its implications to Theorem 17.

> Theorem 18. Let N_1 be a connected, transformerless network containing n consecutively numbered $I\text{-}I_z$ sources and satisfying the three conditions of Theorem 13. If the s, $1 \leq s \leq n$, controlling elements which correspond to the controlled sources u_1, u_2, ..., u_s constitute a circuit of N_1, and if U_s denotes a subset of the first n positive integers which contains u_1, u_2, ..., u_s, then the determinant of condition 3 of Theorem 13 approaches zero in the limit as the values of the transmittance constants corresponding to U_s approach infinity.

Proof. In the limit, the determinant again has a form such that those rows of

$$\left| F_{pl} \quad -\mathcal{A}^{-1} \right|$$

having row indices belonging to U_s contain no terms which involve transmittance constants. In particular, rows $u_1, u_2, \ldots, u_s$ contain no transmittance constants. The denominators of all of the terms f_{ij} of F_{pl} are identical; and the numerators of all of the terms are given by K-linkages. But dividing a K-linkage by the controlling admittance gives the L-linkage which relates the terminal voltage of the controlling element to the appropriate replacement source of N_3. Divide row u_k of

$$\lim_{\substack{\alpha_i \to \infty \\ \alpha_i \, \eta_i \ni U_s}} \left| F_{pl} - \mathcal{A}^{-1} \right|$$

by the nonzero controlling admittance, y_{pu_k}, which corresponds to source u_k, for $k = 1, 2, \ldots, s$. The result is that for every column, the elements of s rows have numerators which are a set of L-linkages corresponding to a circuit, and all of the elements have identical denominators. By Theorem 6, the sum of a set of L-linkages which terminate upon the elements of a circuit is zero. It follows that

$$\sum_{k=1}^{s} \frac{1}{y_{pu_k}} \; f_{u_k j} = 0 \qquad \text{for all } j. \tag{80}$$

Equation 80 states that the rows $u_1, u_2, \ldots, u_s$ of

$$\lim_{\substack{\alpha_i \to \infty \\ \alpha_i \, \eta_i \ni U_s}} \left| F_{pl} - \mathcal{A}^{-1} \right|$$

are linearly dependent in the limit as the transmittance constants corresponding to U_s approach infinity. Therefore, the value of the determinant approaches zero, which proves the theorem.

An explanation based upon physical reasoning is also possible for this theorem. The sum of the L-linkages for any j in Equation 80 is the sum of the voltages which appear across a circuit of admittances in response to replacement source j, when every other replacement source has been properly removed. By Kirchhoff's voltage law, this sum must be zero for every j.

Looking at the problem intuitively, the currents of the controlling elements which form the circuit are forced to approach zero as the transmittance constants approach infinity. This requires that the voltages of the individual circuit elements approach zero, and therefore sum to zero as a result of the transmittance constraints. Thus the controlled sources again independently force a result which was already required by Kirchhoff's law, thereby making the network equations linearly dependent.

Finally, an interesting conclusion may be deduced regarding the locations of the controlling elements when all of the transmittance constants approach infinity.

The equation

$$\lim_{\alpha_1, \alpha_2, \ldots, \alpha_n \to \infty} \left| F_{pl} - \mathcal{A}^{-1} \right| = \left| F_{pl} \right| , \tag{81}$$

and Theorems 17 and 18 imply that

$$\left| F_{pl} - \mathcal{A}^{-1} \right| \neq 0 \tag{82}$$

in the limit as the transmittance constants approach infinity only if N_1 contains no seg consisting only of controlling elements and no circuit consisting only of controlling elements. Since a cut-set is a special case of a seg, these conditions imply that the inequality of Equation 82 holds in the limiting case only if there is neither a cut-set nor a circuit of controlling elements in N_1. But these conditions imply that there is a tree T_a in N_1 such that every controlling element is a chord of T_a, and that there is also a tree T_b which contains every controlling element as a branch. It is interesting to note that these are exactly the constraints found by Blackwell (2) to govern the topological locations of through-across drivers in networks which are described by uniquely solvable network equations.

The theorems and discussions of this chapter have in a sense provided a solution to the solvability problem for some classes of controlled-source networks. More important, however, is that they provide a degree of insight, suggesting areas where further study may lead to solvability conditions of a more fundamental nature. Throughout the solvability studies a close relationship was observed between the expressions obtained for expanded determinants which contain the solvability information and the topological formulas given in the literature which describe a certain subclass of the networks of interest. Closer examination of this relationship leads to Theorem 19 of the next chapter, which relates the signs of the determinants of the non-singular submatrices of A and B and suggests the possibility of deriving topological formulas for networks which contain all of the controlled source types satisfying Postulate P'.

DERIVATION OF TOPOLOGICAL FORMULAS

This chapter demonstrates a new method of deriving topological formulas for networks containing controlled sources. Since a set of element-variable equations rather than the nodal or mesh equations is used, the ordinary restriction which requires that the element-admittance or element-impedance matrix exist is obviated, and all of the controlled-source types satisfying Postulate P', even $I\text{-}I_o$ and $V\text{-}V_o$ sources, are admissible. The general formulas are not developed because problems of term interpretation and problems involving signs require further study. However, a theorem crucial to the sign problem is presented, which, together with a theorem given by Coates (4), make it possible to develop formulas for system determinants of networks with transmittance constants which approach infinity. In the last chapter, this limiting case was shown to be useful as a model for networks containing high-gain operational amplifiers. However, it takes on added importance here since the formula developed constitutes one of the terms in the general expansion. The new formula is used to solve a network problem, and the solution is compared with the solution obtained using nodal equations.

General Derivation Problem

The networks considered here are connected and transformerless and contain only passive elements and controlled sources. Each network is driven by a single fixed current source. Transformers could probably be included by using a controlled-source representation based upon Percival's transformer model (11); however, as a matter of convenience, they are not included here. Also, it will be obvious in the derivation that a fixed voltage source could have been used for the driver with no changes in the derivation procedure.

The starting point is again the network diagram which has been modified to accommodate open-circuit and short-circuit controlling variables by the addition of suitable fixed sources having output functions specified to be zero. A linear graph is constructed from the network diagram such that a one-to-one correspondence exists between the graph edges and network elements, including sources, and between the graph vertices and the nodes of the network.

Because of the nature of the linear graph used in this approach, it is particularly important to note a distinction, pointed out by Mason (6), between two different concepts customarily associated with the work "tree". First, the word tree suggests a geometrical entity, a connected, circuitless subgraph containing all of the vertices of the original graph. But the work "tree" also suggests an algebraic entity, a number given by the product of the admittance weights of the branches of the geometrical tree. In classical development of topological formulas, exemplified by the work of Percival (12) and Coates (4), it was possible to assign an admittance weight to each graph edge, in which case the distinction between the geometrical trees and the algebraic trees was relatively unimportant. But in the linear graph used here, those edges which correspond to sources have no admittance weights. Mason's concept of the algebraic tree will still apply in the discussions that follow, but it should be noted that the algebraic expression which corresponds to a given geometrical tree of v-1 edges will, in general, be a product of less than v-1 admittances. And, it is not always possible to find that particular tree from the graph, for the tree may not be unique.

For a network N_1 satisfying the conditions stated above and containing n I-I_z sources, the general method of approach may be illustrated by writing the network equations in the form

$$\begin{bmatrix} -\mathcal{A}^{-1} & U & 0 & 0 & 0 & 0 & 0 \\ A_c & A_{p1} & A_{p2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_c & B_{p1} & B_{p2} & B_J \\ 0 & -U & 0 & 0 & Y_{p1} & 0 & 0 \\ 0 & 0 & -U & 0 & 0 & Y_{p2} & 0 \end{bmatrix} \begin{bmatrix} I_c \\ I_{p1} \\ I_{p2} \\ V_c \\ V_{p1} \\ V_{p2} \\ V_J \end{bmatrix} = \begin{bmatrix} 0 \\ -A_J J \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{83}$$

where J is the driving source, and the subscripts denoting the variables are the same as in the last chapter.

The determinant of the coefficient matrix of Equation 83 may be developed according to the minor determinants of the first n rows by means of Laplace's expansion. These minors give terms involving combinations of the reciprocals of the transmittance constants taken 0, 1, 2, ..., n at a time, with each such term having an algebraic complement for a coefficient. The expanded determinant then has the form

$$(-1)^{s_1} \begin{vmatrix} A_{p1} & A_{p2} & 0 & 0 & 0 & 0 \\ 0 & 0 & B_c & B_{p1} & B_{p2} & B_J \\ -U & 0 & 0 & Y_{p1} & 0 & 0 \\ 0 & -U & 0 & 0 & Y_{p2} & 0 \end{vmatrix} \cdot \left| \mathcal{A}^{-1} \right| + \ldots +$$

$$+(-1)^{s_2} \begin{vmatrix} A_c & A_{p2} & 0 & 0 & 0 & 0 \\ 0 & 0 & B_c & B_{p1} & B_{p2} & B_J \\ 0 & 0 & 0 & Y_{p1} & 0 & 0 \\ 0 & -U & 0 & 0 & Y_{p2} & 0 \end{vmatrix} \quad , \qquad (84)$$

where s_1 and s_2 are positive integers which arise in Laplace's expansion.

The desired result is to reduce each of the coefficients of Equation 84, by repeated application of Laplace's theorem, to an algebraic expression which is identifiable in terms of the structure of N_1.

Although the procedure is simple enough in principle, some formidable problems arise in the details of deriving and interpreting the algebraic expressions. Therefore the general problem is dropped at this point in favor of a special case.

Networks with Transmittance Constants which Approach Infinity

If all of the transmittance constants of N_1 are assumed to approach infinity, the submatrix $-\mathcal{A}^{-1}$ of Equation 83 vanishes, and the expanded system determinant given by Equation 84 reduces to only one term, the last one. It is the derivation of a topological formula for this determinant which will now occupy our attention.

Derivation of the sign rule

Before beginning the actual derivation, two theorems are presented which are necessary for establishing the signs and interpretations of the various terms arising from the expanded determinant. The second theorem, Theorem 20, was given by Coates (4) and is therefore presented without proof.

Theorem 19. Let G be a connected, directed linear graph having v vertices and e edges. Let A be a given vertex matrix of G. Let B be a given circuit matrix of G, with columns arranged in the same element order as the columns of A. Let $\left|A_{t_i}\right|$ denote the determinant of the non-singular submatrix of A which corresponds to tree t_i of G. Let $\left|B_{t_i}\right|$ denote the determinant of the non-singular submatrix of B which corresponds to the chord set of tree t_i of G. Then for every tree t_i of G,

$$\left|A_{t_i}\right| \cdot \left|B_{t_i}\right| = (-1)^{s_{t_i}} (u)(k'), \tag{85}$$

where s_{t_i} is the number of column interchanges necessary to bring the columns of A_{t_i} to the extreme left of A, k' is a positive integer fixed for a given B, and u is a number, fixed for a given A and B, having a value of either +1 or -1.

Proof. Form the e x e matrix K, where

$$K = \begin{bmatrix} A \\ B \end{bmatrix}. \tag{86}$$

The transpose of K may be written in the form

$$K^T = [A^T \ B^T] \ . \tag{87}$$

Then

$$KK^T = \begin{bmatrix} A \\ B \end{bmatrix} [A^T \ B^T] = \begin{bmatrix} AA^T & AB^T \\ BA^T & BB^T \end{bmatrix}. \tag{88}$$

But it is well-known that

$$AB^T = BA^T = 0; \tag{89}$$

therefore,

$$\begin{bmatrix} A \\ B \end{bmatrix} [A^T\ B^T] = \begin{bmatrix} AA^T & 0 \\ 0 & BB^T \end{bmatrix}. \tag{90}$$

Since the determinant of the product of two square matrices equals the product of the determinants of the matrices, and since the determinant of the transpose of a square matrix equals the determinant of the matrix, taking the determinant of both sides of Equation 90 gives

$$\begin{vmatrix} A \\ B \end{vmatrix}^2 = \left| AA^T \right| . \left| BB^T \right| . \tag{91}$$

But, according to Trent (18),

$$\left| AA^T \right| = n_t \quad , \tag{92}$$

where n_t is the number of trees in G.

Also, according to Cederbaum (3), every non-singular submatrix of B has a determinant given by $\pm$ k', where k' is an integer, fixed for a given B. Since every non-singular submatrix of B corresponds to the chord set of a tree of G, by the Binet-Cauchy theorem,

$$\left| BB^T \right| = n_t \quad (k')^2. \tag{93}$$

Combining Equations 91, 92, and 93 then gives

$$\begin{vmatrix} \mathbf{A} \\ \\ B \end{vmatrix}^2 = n_t^2 \; (k')^2, \tag{94}$$

or

$$\begin{vmatrix} \mathbf{A} \\ \\ B \end{vmatrix} = \pm \; n_t \; (k'). \tag{95}$$

Now the determinant on the left of Equation 95 may be developed by Laplace's expansion according to minors of the first v-1 rows. By Theorem 3, a minor determinant formed by choosing some set of v-1 columns of **A** is nonzero if and only if these columns correspond to some tree t_i. But crossing out the rows and columns of this minor leaves a complementary minor consisting of a submatrix of B having e-v+1 rows and columns. Furthermore, the columns of this complementary minor correspond exactly to the chord set of tree t_i; and therefore the complementary minor is also nonzero. Thus, ignoring the vanishing terms, Equation 95 becomes

$$\sum_{i=1}^{n_t} (-1)^{s_{t_i}} \left| A_{t_i} \right| \cdot \left| B_{t_i} \right| = \pm \; n_t \; (k'), \tag{96}$$

where s_{t_i} is the number of column interchanges necessary to bring the columns of A_{t_i} into the leading position of A. But since

$$\left| B_{t_i} \right| = \pm \, k' \tag{97}$$

and

$$\left| A_{t_i} \right| = \pm 1 \tag{98}$$

for every tree t_i in G, the left side of Equation 96 is a sum of n_t terms, each term having the form $\pm$ k'. Therefore, for the equality of Equation 96 to hold, it is both necessary and sufficient that every term on the left side have the same sign, and that this sign agree with the sign on the right side. Thus it has been proven that

$$(-1)^{s_{t_i}} \left| A_{t_i} \right| \cdot \left| B_{t_i} \right| = u(k') \tag{99}$$

for every tree t_i, where u is either a +1 or a -1. Equation 99 is equivalent to Equation 85, which completes the proof.

Before going on to Coates' theorem, it might be pointed out that Equation 95 provides a new formula for determining the number of trees in a connected graph. The integer k' which appears in Equation 99 is the same as the k' in Equation 67. As mentioned before, k' is one if the circuits of B are fundamental circuits or mesh circuits of a planar graph. In these cases, the determinant gives the number of trees directly; otherwise k' must be determined by evaluating the determinant of one of the nonsingular submatrices of B.

Unfortunately, Coates' theorem involves some rather esoteric concepts which cannot be grasped without a careful study of a number of definitions. These definitions, most of which were given by Coates (4), are listed below together with a few clarifying statements.

1. For a graph of v vertices and e edges, an <u>incidence matrix</u> $A_a = [a_{p_i \epsilon j}]$ consists of v rows and e columns such that

$a_{p_i \epsilon j} = 1$ if edge ϵ_j is incident at vertex p_i and is oriented away from vertex p_i,

$a_{p_i \epsilon j} = -1$ if edge ϵ_j is incident at vertex p_i and is oriented toward vertex p_i, and

$a_{p_i \epsilon j} = 0$ if edge ϵ_j is not incident at vertex p_i.

2. Any v-1 rowed submatrix of A_a is called a <u>vertex matrix</u> of the graph and is denoted by A.

3. The vertex symbol of row v of A_a, which is excluded from A, is the <u>reference</u> of A.

4. Corresponding to the rows of a given vertex matrix is a <u>vertex sequence</u> $\{(p_i)_k\}$ such that the k th member of the sequence is the symbol for the k th row of A.

5. Similarly, an <u>edge sequence</u> $\{(\epsilon_j)_k\}$ is associated with the columns of A.

6. An <u>oriented path</u> with initial vertex p_i and final vertex p_j is denoted by $P\{p_i, p_j\}$. The orientation direction of a path is from the initial vertex to the final vertex. In particular, the oriented path which begins at p_i and terminates at the reference vertex is denoted by $P\{p_i, p_v\}$.

Figure 13 shows two oriented paths of a connected graph, which has reference vertex p_v. This figure will also be helpful in understanding the following two definitions.

Let T denote a connected circuitless graph or subgraph of v vertices with vertex matrix A_T and corresponding edge and vertex sequences $\{(\epsilon_j)_k\}$

and $\{(p_i)_k\}$ respectively.

7. If ϵ_j belongs to T and is incident to vertices p_h and p_k, then p_k is the <u>left vertex</u> of ϵ_j provided $P\{p_k, p_v\}$ of T does not include ϵ_j.

8. Similarly, p_h is the <u>right vertex</u> of ϵ_j provided $P\{p_h, p_v\}$ of T includes ϵ_j.

Because T has exactly v-1 edges and exactly one path $P\{p_i, p_v\}$ for each $p_i \neq p_v$, it is evident that each vertex of T, other than p_v, is the right vertex of exactly one edge and that each edge of T has both a right and a left vertex.

9. That edge for which p_i, $p_i \neq p_v$, is the right vertex is called the <u>principal edge</u> of p_i. Or, stated less elegantly but more lucidly, the principal edge of vertex p_i is the first edge encountered in traversing the tree path from p_i to p_v.

10. Associated with each p_i, $p_i \neq p_v$, of T is a <u>vertex weight</u> νp_i such that $\nu p_i = 1$ if the orientation of the principal edge of p_i agrees with the orientation of $P\{p_i, p_v\}$, and $\nu p_i = -1$ if the orientation of the principal edge of p_i is opposite to the orientation of $P\{p_i, p_v\}$.

11. If $\{(p_i)_k\}$ is a vertex sequence of T, the <u>principal edge sequence</u> of T with respect to $\{(p_i)_k\}$ is the sequence of edges $\{(\epsilon_R)_k\}$ such that each $(\epsilon_R)_k$ is the principal edge of $(p_i)_k$.

Since the admissible controlled sources satisfy Postulate P', the graphs considered here contain exactly one edge corresponding to each controlled source and exactly one edge corresponding to the controlling element of that source.

12. The term <u>c^1-edge</u> denotes an edge which corresponds to a controlled source. The term <u>c^2-edge</u> represents the edge corresponding to the con-

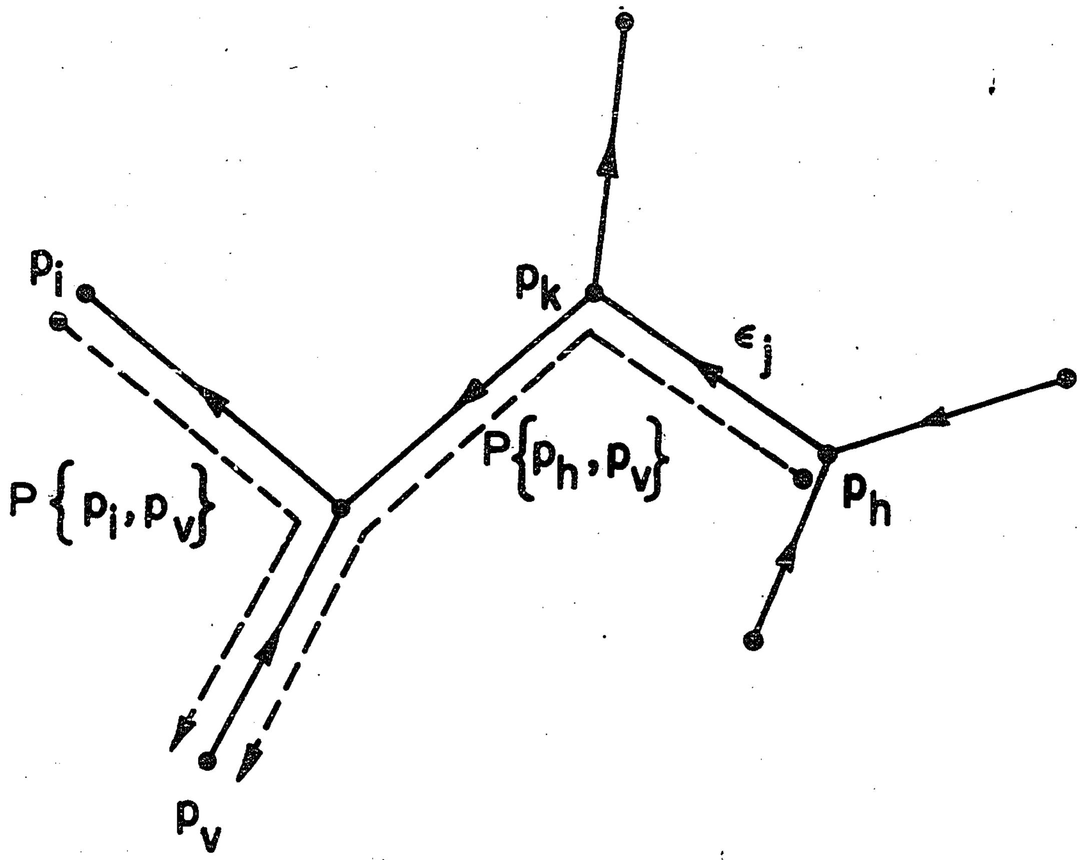


Figure 13. Oriented paths of a connected graph

trolling element of that controlled source.

13. The term edge pair refers collectively to a c^1-edge and its corresponding c^2-edge.

14. The term c-edge denotes a single member of an edge pair.

15. A single c-edge of a tree T is a member ϵ_j of an edge pair such that ϵ_j belongs to T but the other member of the edge pair does not belong to T.

16. If T is a subgraph of G, let β denote the collection of single c-edges of T. Associated with T is a connected circuitless subgraph T_β, called the signature of T, which is obtained from T by the following process. Let $\{t_k\}$ denote the set of connected subgraphs of T which result when the line segments of β are deleted from T. Although some members of $\{t_k\}$ may be isolated vertices, each t_k contains exactly one right vertex of some ϵ_j of β, except for the t_k which contains the reference vertex. Let each t_k be shrunk to a single vertex, and label each vertex with the symbol p_k. The resultant is the connected circuitless graph T whose edges and corresponding right vertex weights are the same as those of T which belong to β.

Figure 14a shows a tree t in which the single c-edges are assumed to be ϵ_4 and ϵ_7. That is β is the set (ϵ_4, ϵ_7). For this tree, the set of connected subgraphs, $\{t_k\} = \{a,b,v\}$, is shown in Figure 14b. The signature, T_β, of T is shown in Figure 14c.

17. Corresponding to each graph G there is a companion graph G' obtained from G by interchanging the edge symbols of the two members of each edge pair. Graphs G and G' are called a graph pair. Furthermore, for each vertex matrix A of G, there is a corresponding vertex matrix A' of G' such that the edge and vertex sequences of the two are identical. From the

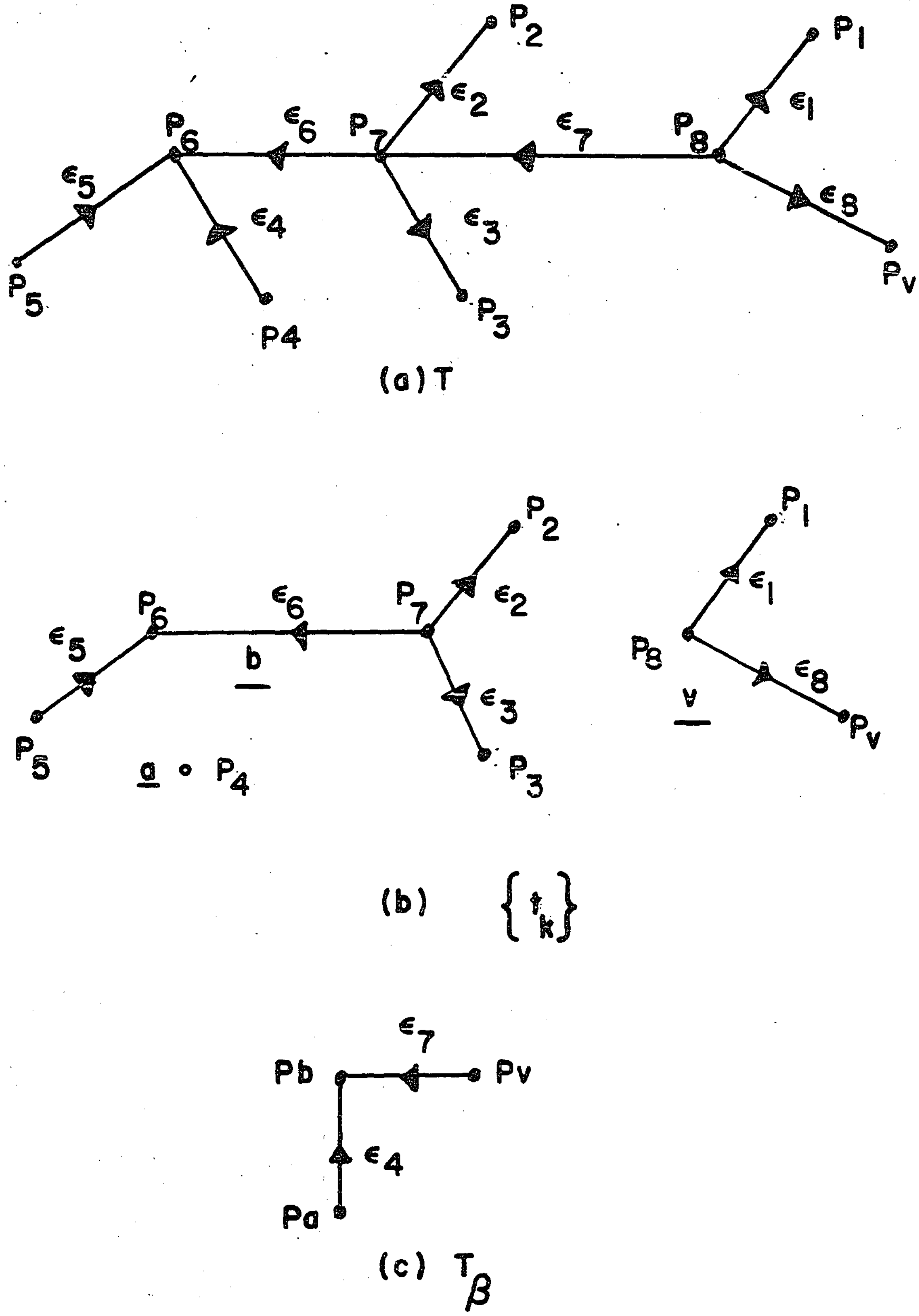


Figure 14. A tree T, its subgraphs and its signature

definition of G'. it follows that A' may be obtained from A by interchanging every pair of columns of A which corresponds to an edge pair.

These ideas are clarified by means of the following example. A graph pair, G and G', are shown in Figure 15, where (1,5) and (3,2) are assumed to be two edge pairs. The vertex matrices having vertex sequence abc and edge sequence 231465 are given by

$$A = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{cccccc} 2 & 3 & 1 & 4 & 6 & 5 \end{array} \\ \begin{bmatrix} 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & -1 \end{bmatrix} \end{array} \qquad (100)$$

and

$$A' = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{cccccc} 2 & 3 & 1 & 4 & 6 & 5 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 1 & -1 & -1 & 0 & 0 & 0 \end{bmatrix} \end{array} . \qquad (101)$$

18. A tree set t of G is a set of v-1 edges, the corresponding G subgraph of which is a tree. A <u>complete tree set</u> of G is a set of v-1 edges which is a tree of both G and G'.

For example, in Figure 15, 265 is a tree of both G and G', and therefore constitutes a complete tree of G. But 126 is not a complete tree of G because 126 of G' contains a circuit.

It will be helpful later to think of the complete tree of a graph G in terms of the matrix

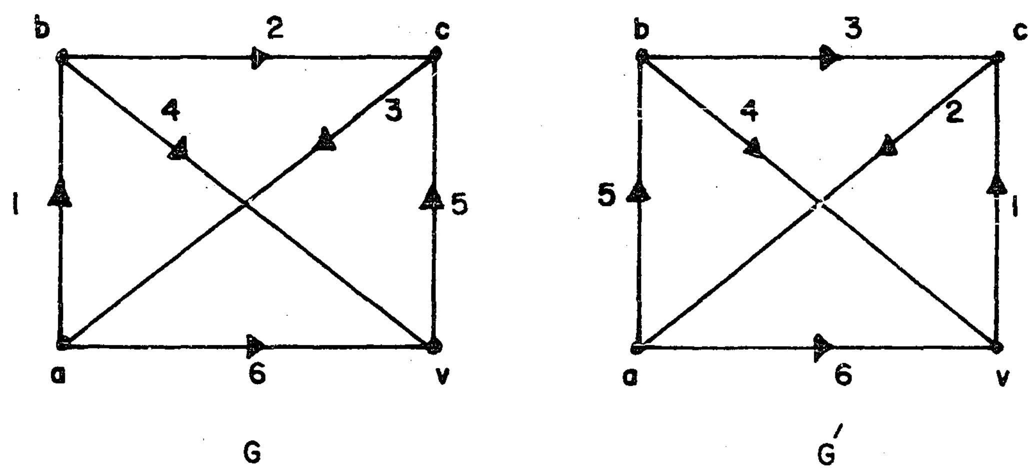


Figure 15. Graph pair with edge pairs (1,5) and (3,2)

$$D = \begin{bmatrix} A \\ \\ A' \end{bmatrix} . \tag{102}$$

If a set of v-1 columns of D are selected such that the minor $|A_{t_i}|$ formed from these columns and the first v-1 rows of D is nonzero, then these columns correspond to some tree t_i of G. But in selecting these columns, we have also automatically selected that minor, $|A'_{t_i}|$ of A', which corresponds to the same v-1 edges of G', and which may or may not constitute a tree of G'. If this subgraph is a tree of G', then $|A'_{t_i}|$ is nonzero, and the set of edges which correspond to the selected columns constitute a complete tree of G.

19. Let T and T' denote the subgraphs of G and G' respectively which correspond to some complete tree $\mathcal{T}$. Let $\mathcal{T}$ include a set δ of N edge pairs and a set β of single c-edges. Let T_β and T'_β denote the signatures of T and T' such that corresponding vertices of T_β and T'_β bear the same label; where those vertices which correspond are those whose corresponding subgraphs of T and T' contain the same vertices. Let $\{\epsilon_{\beta k}\}$ and $\{\epsilon'_{\beta k}\}$ denote the principal edge sequences of T_β and T'_β for an arbitrary vertex sequence $\{(p_i)_k\}$. Let n denote the number of pair permutations necessary to transform $\{\epsilon_{\beta k}\}$ into $\{\epsilon'_{\beta k}\}$. Let ν_{p_i} and ν'_{p_i} denote the vertex weights of p_i of T_β and T'_β respectively. Let q denote the <u>weight</u> of $\mathcal{T}$ which is defined as

$$q = (-1)^{N+n} \prod_{i \,\eta\, p_i \ni T_\beta} \nu_{p_i} \nu'_{p_i}, \tag{103}$$

where $i \;\eta\; p_i \ni T_\beta$ means all i such that vertex p_i belongs to T_β. For the case where Υ includes no single edge pairs (β is empty), define

$$q = (-1)^{N}. \tag{104}$$

This definition is clarified by the following example. The subgraphs T and T' corresponding to the complete tree 256 of Figure 15 are shown in a and b of Figure 16, and the signatures T_β and T'_β are shown in c and d. For this example, T contains no edge pairs, therefore δ is an empty set, and N is zero. The single c-edges of β are edges 2 and 5. Then for vertex sequence $\{wxv\}$, the edge sequences are

$$\{\epsilon_{\beta k}\} = \{25\} \tag{105}$$

$$\{\epsilon'_{\beta k}\} = \{52\} .$$

Then n = 1, since one pair permutation suffices to transform $\{\epsilon_{\beta k}\}$ into $\{\epsilon'_{\beta k}\}$. The vertex weights are $\nu_{pw} = +1$, $\nu_{px} = -1$, $\nu'_{pw} = -1$, and $\nu'_{px} = +1$, which gives the weight of the complete tree

$$q = (-1)^{0+1} \; [(+1)(-1)] \; [(-1)(+1)] = -1. \tag{106}$$

With this list of definitions and statements as background, Coates' theorem may be stated quite simply.

> Theorem 20. (Coates' Theorem). If the trees of G and G' which correspond to a complete tree Υ_i have vertex matrices A_{t_i} and A'_{t_i} then

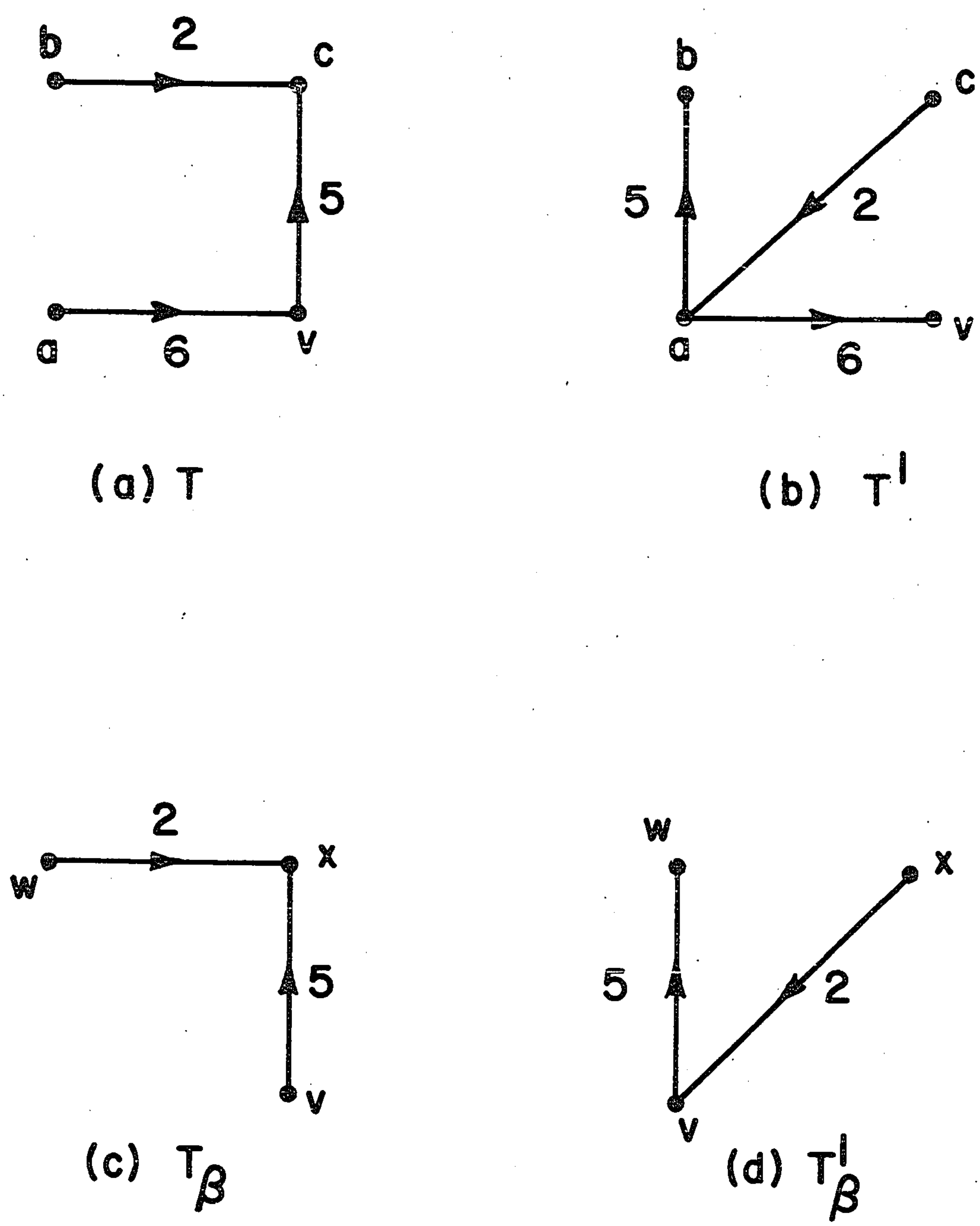


Figure 16. Complete tree and its two signatures

$$\left| A_{t_i} \right| \cdot \left| A'_{t_i} \right| = q_i , \tag{107}$$

where q_i is the weight of τ_i.

Now sufficient information is available to derive a sign rule which will be needed in the topological formula derivation. Let A and B be the vertex and circuit matrices of the linear graph G, with the columns of A and B arranged in the same element order. Let A' and B' be vertex and circuit matrices formed from A and B by interchanging the columns which correspond to the edge pairs of G. Now consider the Laplace's expansion of the determinant

$$\underline{D} = \begin{vmatrix} A \\ B' \end{vmatrix} \tag{108}$$

according to nonzero minors of the first v-1 rows of $\underline{D}$. Each nonzero term μ_i in the expansion will be of the form

$$\mu_i = (-1)^{s_{t_i}} \left| A_{t_i} \right| \cdot \left| B'_{t_i} \right| , \tag{109}$$

where the columns of A_{t_i} correspond to the set of elements τ_i of a tree t_i of G, the columns of B'_{t_i} correspond to the chord set of a tree of G' which consists of the set of edges τ_i, and s_{t_i} is the number of column interchanges necessary to bring A_{t_i} to the leading position of A.

As shown in Equations 97 and 98, an algebraic sign is associated with both $\left| A_{t_i} \right|$ and $\left| B'_{t_i} \right|$. Finding a relationship between these two signs is the next objective.

Theorem 19, which applies to any given A and B of a connected, directed graph, holds for A' and B' of G'. Therefore, it is true that if

$\left| A'_{t_i} \right|$ is nonzero,

$$\left| A'_{t_i} \right| \cdot \left| B'_{t_i} \right| = (1)^{s'_{t_i}} (u')(k'), \tag{110}$$

where s'_{t_i} is the number of column interchanges necessary to bring the columns of A'_{t_i} into the leading position of A'. But, by Theorem 20,

$$\left| A'_{t_i} \right| = q_{t_i} \left| A_{t_i} \right| , \tag{111}$$

since $\left| A_{t_i} \right| = \pm 1$. Substituting Equation 111 into Equation 110 gives

$$\left| A_{t_i} \right| \cdot \left| B'_{t_i} \right| = (1)^{s'_{t_i}} (u')(k')\, q_{t_i} , \tag{112}$$

since $q_{t_i} = \pm 1$. Equation 112 holds for every tree t_i which is a complete tree of G. Also, since s_{t_i} is the number of column interchanges necessary to bring the columns of A_{t_i} into the leading position of A, and since s'_{t_i} is the number of interchanges necessary to bring the columns of A'_{t_i} into the leading position of A', it follows from inspection of Equation 102 that

$$s_{t_i} = s'_{t_i} . \tag{113}$$

Now these results will be used to derive a topological formula for the system determinant of Equation 83 for the limiting case of the transmittance constants approaching infinity.

Derivation of topological formulas

In order to eliminate some of the bookkeeping involved in keeping track of the signs, the rows and columns of the coefficient matrix of Equation 83 are rearranged and some rows multiplied by minus one such that, with no loss of generality, the rearranged coefficient matrix may be written in the form

$$C = \begin{bmatrix} 0 & 0 & A_c & 0 & 0 & A_{p1} & A_{p2} \\ B_c & B_{p2} & 0 & B_J & B_{p1} & 0 & 0 \\ 0 & 0 & -\mathcal{L}^{-1} & 0 & 0 & U & 0 \\ 0 & 0 & 0 & 0 & -Y_{p1} & U & 0 \\ 0 & -Y_{p2} & 0 & 0 & 0 & 0 & U \end{bmatrix} . \tag{114}$$

Now as the transmittance constants approach infinity, Equation 114 becomes

$$C_\infty = \begin{bmatrix} 0 & 0 & A_c & 0 & 0 & A_{p1} & A_{p2} \\ B_c & B_{p2} & 0 & B_J & B_{p1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & U & 0 \\ 0 & 0 & 0 & 0 & -Y_{p1} & U & 0 \\ 0 & -Y_{p2} & 0 & 0 & 0 & 0 & U \end{bmatrix} . \tag{115}$$

Since Y_{p1} and Y_{p2} are diagonal, C_∞ of Equation 115 may be transformed by elementary column operations into

$$\underline{C}_\infty = \begin{bmatrix} 0 & A_{p2}\,Y_{p2} & A_c & 0 & A_{p1}\,Y_{p1} & A_{p1} & A_{p2} \\ B_c & B_{p2} & 0 & B_J & B_{p1} & 0 & 0 \\ 0 & 0 & 0 & 0 & Y_{p1} & U & 0 \\ 0 & 0 & 0 & 0 & 0 & U & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & U \end{bmatrix}, \tag{116}$$

which has the same determinant as c_∞. But an elementary result of Laplace's theorem is

$$\left|\underline{C}_\infty\right| = \begin{vmatrix} 0 & A_{p2}\,Y_{p2} & A_c & 0 \\ & & & \\ B_c & B_{p2} & 0 & B_J \end{vmatrix} p(Y_{p1}) , \tag{117}$$

where p (Y_{p1}) denotes the product of the admittances of Y_{p1}.

Now it is necessary to develop the determinant on the right side of Equation 117 according to minors of the first v-1 rows. Since Y_{p2} is diagonal, the product $A_{p2}Y_{p2}$ is simply the matrix A_{p2} with every column multiplied by one of the nonzero admittances of Y_{p2}. Thus every minor from the first v-1 rows is a determinant of order v-1 taken from $[A_{p2}\ A_c]$ multiplied by those admittances of Y_{p2} which correspond to the columns selected from A_{p2} to form the minor. Since the columns of every nonzero determinant from $[A_{p2}\ A_c]$ corresponds to some tree of G, the admittance product from Y_{p2} which is associated with each term is the product of all of the admittances included in this tree. Every nonzero term of the expansion of Equation 117 then consists of the product of three quantities: p (Y_{p1}), a term from the expansion of the determinant

$$D = \begin{vmatrix} 0 & A_{p2} & A_c & 0 \\ \\ B_c & B_{p2} & 0 & B_J \end{vmatrix} , \tag{118}$$

and a product of the admittances of those elements which correspond to the columns of A_{p2} used in forming the minor from the first v-1 rows.

If the matrices A and B of G are defined by

$$A = [A_{p1}\ A_{p2}\ A_c\ A_J] \tag{119}$$

and

$$B = [B_{p1}\ B_{p2}\ B_c\ B_J] \tag{120}$$

then A' and B' are given by

$$A' = [A_c\ A_{p2}\ A_{p1}\ A_J] \tag{121}$$

and

$$B' = [B_c\ B_{p2}\ B_{p1}\ B_J] \ . \tag{122}$$

Thus

$$\underline{D} = \begin{vmatrix} A_{p1} & A_{p2} & A_c & A_J \\ \\ B_c & B_{p2} & B_{p1} & B_J \end{vmatrix} , \tag{123}$$

which is a partitioned form of Equation 108. From the discussion of Equation 108, it follows that the expansion of $\underline{D}$ according to the minors of the first v-1 rows is a sum of nonzero terms which corresponds to the complete tree set of G. Comparison of Equations 118 and 123 shows that the terms of the expansion of Equation 118 are simply a subset of the terms arising from the expansion of $\underline{D}$ of Equation 123, or a subset of the complete tree set of G. Therefore, Laplace's expansion of Equation 118 may be written in the form

$$D = \sum_{i=1}^{n} (-1)^{s_{t_i}} \left| A_{t_i} \right| \cdot \left| B'_{t_i} \right| . \tag{124}$$

Inspection of Equation 118 shows that every nonzero product $\left| A_{t_i} \right| \cdot \left| B'_{t_i} \right|$ in Equation 124 implies that N_1 contains a complete tree t_i of G, such that t_i contains every controlled source of N_1, and such that the driver, J, and all controlling elements are in the chord set of t_i. For by inspection of Equation 118, these are seen to be necessary and sufficient conditions for forming a nonzero minor from $[A_{p2}\ A_c]$ when the additional constraint imposed by the zero beneath submatrix A_c in Equation 118 is taken into consideration. Let $\underline{n}$ be the number of complete trees t_i in N_1 which satisfy these conditions, and let $p_y(t_i)$ represent the product of the admittances contained in t_i. Then from Equations 117, 118, 124, and the observations made about the product $A_{p2}\ Y_{p2}$ of Equation 117 we may conclude that

$$\left| \underline{C}_{\infty} \right| = \sum_{i=1}^{n} (-1)^{s_{t_i}} \left| A_{t_i} \right| \cdot \left| B'_{t_i} \right| p_y(t_i)\ p(Y_{p1}) . \tag{125}$$

Using Equations 112 and 113, Equation 125 may be written in the form

$$\left| \underline{C}_{\infty} \right| = (u')(k') \left\{ \sum_{i=1}^{n} p_y(t_i)\, q_i \right\} p(Y_{pl}), \tag{126}$$

which is the desired topological formula for the system determinant of Equation 83 in the limiting case of the transmittance constants approaching infinity.

The mechanics of using Equation 126 in an actual problem involve two procedures, (a) finding the suitable set of complete trees t_i and (b) determining the proper sign, q_i, to be associated with the algebraic expression for each t_i.

The task of finding those complete trees which correspond to terms of Equation 126 may be accomplished in a simple and straightforward manner. Since only those trees of having the driver and all n controlling elements in their chord sets need be considered, it follows from Theorem 5 that the branches of G corresponding to these elements may be removed from G before the search begins. The resulting graph, $\underline{G}$, containing n + 1 few edges than G, still contains all of the desired trees. But each t_i must also contain every branch which corresponds to one of the n controlled sources. By Theorem 5, it is possible to find these trees using an additional reduction of the graph. Every edge which corresponds to a controlled source is removed from $\underline{G}$, and the two vertices to which the edge was previously incident are identified. Any self-loops produced by this process are removed. The result is a graph, $\overline{\underline{G}}$, containing at least 2n + 1 fewer edges than G. Now each tree of $\overline{\underline{G}}$ is found. To the set of edges of each of these trees are added the n c^1 edges which correspond to the controlled sources. The result is a set of trees of G containing branches corresponding to every controlled source of N_1 but containing no branches which correspond to the controlling elements or the driver. The trees of this set which are also trees of G' constitute the trees t_i, ($i = 1, 2, \ldots, \underline{n}$).

Finding the signs for the allowable complete trees t_i requires that the signature for every tree t_i of G and its corresponding tree in G' be constructed. Then each q_i is calculated by Equation 103 using the information contained in the pair of signatures.

Now a formula will be derived for an n-source $I\text{-}V_y$ network. The system determinant for a connected, transformerless network N_1 containing n $I\text{-}V_y$ sources and driven by a current source J may be arranged in the form

$$|C| = \begin{vmatrix} 0 & 0 & A_c & 0 & 0 & A_{p1} & A_{p2} \\ B_c & B_{p2} & 0 & B_J & B_{p1} & 0 & 0 \\ 0 & 0 & -\mathcal{A}^{-1} & 0 & U & 0 & 0 \\ 0 & 0 & 0 & 0 & -Y_{p1} & U & 0 \\ 0 & -Y_{p2} & 0 & 0 & 0 & 0 & U \end{vmatrix} . \tag{127}$$

In the limit, as the transmittance constants approach infinity, $\mathcal{A}^{-1}$ vanishes, leaving

$$|C_\infty| = \begin{vmatrix} 0 & 0 & A_c & 0 & 0 & A_{p1} & A_{p2} \\ B_c & B_{p2} & 0 & B_J & B_{p1} & 0 & 0 \\ 0 & 0 & 0 & 0 & U & 0 & 0 \\ 0 & 0 & 0 & 0 & -Y_{p1} & U & 0 \\ 0 & -Y_{p2} & 0 & 0 & 0 & 0 & U \end{vmatrix} , \tag{128}$$

which may be transformed by elementary column operations to

$$\left| \underline{C}_{\infty} \right| = \begin{vmatrix} 0 & A_{p2}\,Y_{p2} & A_c & 0 & A_{p1}\,Y_{p1} & A_{p1} & A_{p2} \\ B_c & B_{p2} & 0 & B_J & B_{p1} & 0 & 0 \\ 0 & 0 & 0 & 0 & U & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & U & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & U \end{vmatrix}, \tag{129}$$

or,

$$\left| \underline{C}_{\infty} \right| = \begin{vmatrix} 0 & A_{p2}\,Y_{p2} & A_c & 0 \\ & & & \\ B_c & B_{p2} & 0 & B_J \end{vmatrix}. \tag{130}$$

Comparison of Equation 130 with Equation 117 shows that the formula for the case considered here differs from that of the $I\text{-}I_z$ case only in the absence of the factor $p(Y_{p1})$. Therefore, by the argument given previously, the determinant of Equation 130 is given by

$$\left| \underline{C}_{\infty} \right| = (u')(k') \left\{ \sum_{i=1}^{n} p_y(t_i) \quad q_i \right\} . \tag{131}$$

The system equation for the n-source $V\text{-}V_o$ case is

$$\begin{bmatrix} 0 & 0 & A_c & 0 & 0 & A_{p2} \\ B_c & B_{p2} & 0 & B_J & B_j & 0 \\ -\mathcal{K}^{-1} & 0 & 0 & 0 & U & 0 \\ 0 & -Y_{p2} & 0 & 0 & 0 & U \end{bmatrix} \begin{bmatrix} V_c \\ V_{p2} \\ I_c \\ V_J \\ V_j \\ I_{p2} \end{bmatrix} = \begin{bmatrix} -A_j I_j - A_J J \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{132}$$

where the subscript j denotes the replacement current sources whose output I_j is specified to be zero. Equation 132 contains no matrix Y_{p1} because N_1 contains no controlling elements which are admittances.

In the limit, as the transmittance constants approach infinity, the system determinant of Equation 132 becomes

$$\left| C_\infty \right| = \begin{vmatrix} 0 & 0 & A_c & 0 & 0 & A_{p2} \\ B_c & B_{p2} & 0 & B_J & B_j & 0 \\ 0 & 0 & 0 & 0 & U & 0 \\ 0 & -Y_{p2} & 0 & 0 & 0 & U \end{vmatrix}, \tag{133}$$

which is equivalent to

$$\left| \underline{C}_\infty \right| = \begin{vmatrix} 0 & A_{p2}\,Y_{p2} & A_c & 0 & 0 & A_{p2} \\ B_c & B_{p2} & 0 & B_J & B_j & 0 \\ 0 & 0 & 0 & 0 & U & 0 \\ 0 & 0 & 0 & 0 & 0 & U \end{vmatrix}. \tag{134}$$

But the determinant of Equation 134 is given by

$$\left| \underline{C}_\infty \right| = \begin{vmatrix} 0 & A_{p2}\,Y_{p2} & A_c & 0 \\ & & & \\ B_c & B_{p2} & 0 & B_J \end{vmatrix}, \tag{135}$$

which is the same as Equation 130. Thus the formula for the V-V_o case is also given by Equation 131.

Although all of the derivations are not presented here, it is possible to prove that when the transmittances approach infinity, the system determinant for any connected, transformerless, n-source network is given by one of two topological formulas, either that of Equation 126, or that of Equation 131. The formula of Equation 126, which contains the factor $p\ (Y_{p1})$, applies to $I\text{-}I_z$ and $V\text{-}I_z$ networks; that is, to those networks in which controlling currents flow through admittances. The formula of Equation 131 applies to the other six n-source network types.

Now a topological formula will be given for a more general network. Let N_1 be a connected, transformerless network containing all types of short-circuit and open-circuit controlled sources. The type of controlled source, the number of sources of that type, and the submatrices of **A** and B corresponding to controlled and controlling variables of N_1 are given in Table 1.

Table 1. Notation for network having short and open-circuit controlling elements

Source type	Number of sources	Submatrices for sources	Submatrices for controlling elements
$I\text{-}I_o$	n_{c1}	A_{c1}, B_{c1}	A_{e1}, B_{e1}
$V\text{-}I_o$	n_{c2}	A_{c2}, B_{c2}	A_{e2}, B_{e2}
$I\text{-}V_o$	n_{c3}	A_{c3}, B_{c3}	A_{j3}, B_{j3}
$V\text{-}V_o$	n_{c4}	A_{c4}, B_{c4}	A_{j4}, B_{j4}

By writing the equation for this network in general form and using the methods of the preceding cases, it is possible to prove that the system determinant is given by

$$\left| \underline{C}_{\infty} \right| = \begin{vmatrix} 0 & 0 & 0 & 0 & A_{p2}Y_{p2} & A_{c1} & A_{c2} & A_{c3} & A_{c4} & 0 \\ B_{c1} & B_{c2} & B_{c3} & B_{c4} & B_{p2} & 0 & 0 & 0 & 0 & B_J \end{vmatrix}, \qquad (136)$$

which is simply a partitioned form of Equation 130. Therefore the topological formula of Equation 131 applies to the system determinant of this network.

Finally, the system equation may be written for a connected, transformerless network which has controlled-source types, numbers of controlled sources and submatrices of A and B as given in Table 2.

Table 2. Notation for general admittance-controlled network

Source type	Number of sources	Submatrices for sources	Submatrices for controlling elements
$I\text{-}I_z$	n_{c5}	A_{c5}, B_{c5}	A_{p5}, B_{p5}
$V\text{-}I_z$	n_{c6}	A_{c6}, B_{c6}	A_{p6}, B_{p6}
$I\text{-}V_y$	n_{c7}	A_{c7}, B_{c7}	A_{p7}, B_{p7}
$V\text{-}V_y$	n_{c8}	A_{c8}, B_{c8}	A_{p8}, B_{p8}

By using the methods of the previous proofs, it can be shown that the system determinant for this network is given by

$$\left| \underline{C}_{\infty} \right| = \begin{vmatrix} 0 & 0 & 0 & 0 & A_{p2}\,Y_{p2} & A_{c5} & A_{c6} & A_{c7} & A_{c8} & 0 \\ B_{c5} & B_{c6} & B_{c7} & B_{c8} & B_{p2} & 0 & 0 & 0 & 0 & B_J \end{vmatrix} \cdot \left| Y_{p5} \right| \cdot \left| Y_{p6} \right| \qquad (137)$$

where Y_{p5} and Y_{p6} are the diagonal admittance matrices of order n_{c5} and n_{c6}, respectively, which correspond to products of the controlling admittances for the current controlled sources of the network. The topological formula for the system determinant in this general case is then given by

$$\left| \underline{C}_{\infty} \right| = u'k' \left\{ \sum_{i=1}^{n} p_y(t_i)\, q_i \right\} p(Y_{p5})\, p(Y_{p6}). \tag{138}$$

If $p(Y_p)$ is defined to be the product of all of the controlling admittances for current-controlled sources, and if the value of $p(Y_p)$ is defined to be one when the network contains no current-controlled sources, then

$$\left| \underline{C}_{\infty} \right| = u'k' \left\{ \sum_{i=1}^{n} p_y(t_i)\, q_i \right\} p(Y_p) \tag{139}$$

is a general topological formula for the system determinant of an admittance-controlled network when all transmittance constants approach infinity.

<u>An example</u>

The system determinant for the network shown in Figure 17 will be calculated using the new topological formula. The graphs G and G' for the network are shown in Figure 18, where the controlled sources are represented by branches 2 and 6 of G, and the edge pairs are (2,1) and (6,3).

Figure 19a shows the graph $\underline{G}$ formed from G by removing edges 1, 3, and J. Figure 19b shows the graph $\overline{\underline{G}}$ formed from $\underline{G}$ by removing edges 2 and 6 and identifying the pairs of vertices to which these edges had been incident. The set of trees of $\overline{G}$ is given by $(4 + 5 + 7 + 8)$, therefore the acceptable sub-set of the trees of $\underline{G}$ is

$$T = 264 + 265 + 267 + 268. \tag{140}$$

From these trees must be selected the ones which are also trees of G'. From Figure 18 it may be verified that these are 265 and 268. These trees contain only edges 5 and 8, respectively, which correspond to admittances of N_1, therefore $p_y(t_i)$ will contain the admittances y_5 and y_8. Next, the signs of the two terms in the sum must be calculated.

Figure 20a shows the trees T_{1G} and T'_{1G} of G and G', respectively, which correspond to the term 268. Figure 18b shows the signatures, $T_{1\beta}$ and $T'_{1\beta}$, of these trees. For vertex sequence $\{p_a p_b\}$, the principal edge sequences from $T_{1\beta}$ and $T'_{1\beta}$ are $\{26\}$ and $\{62\}$, respectively, so that n = 1. The product of vertex weights is given by

$$\nu_{pa}\,\nu'_{pa}\,\nu_{pb}\,\nu'_{pb} = (-1)(-1)(+1)(-1) = -1. \tag{141}$$

Since N = 0, we have

$$q_{T_1} = (-1)^{0+1}\,(-1) = +1. \tag{142}$$

Figure 21a shows the trees T_{2G} and T'_{2G} which correspond to the term 265, and Figure 21b shows the corresponding signatures. From these,

$$q_{T_2} = (-1)^{0+0}\,(-1)(+1)(-1)(-1) = -1. \tag{143}$$

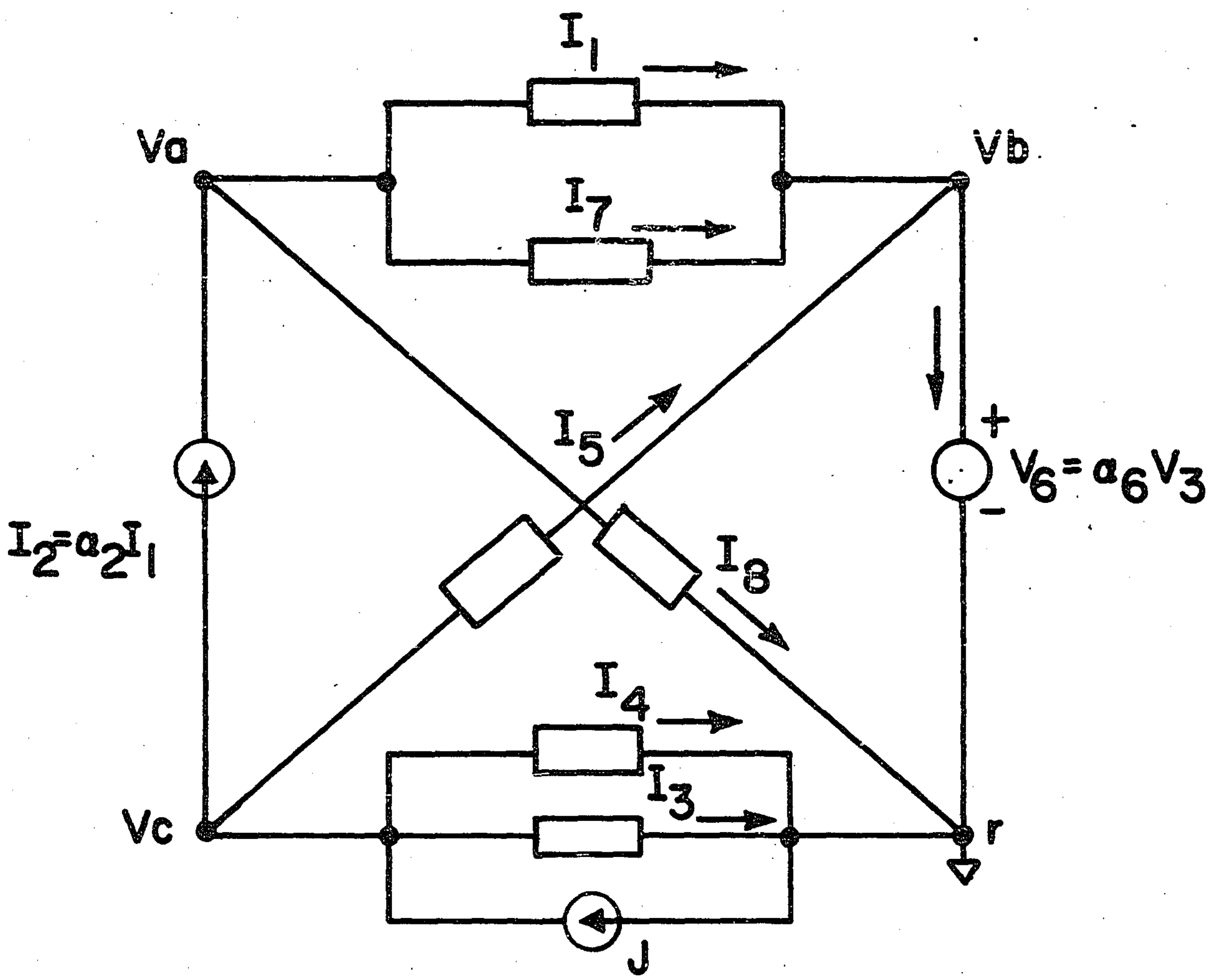


Figure 17. Example of a controlled-source network

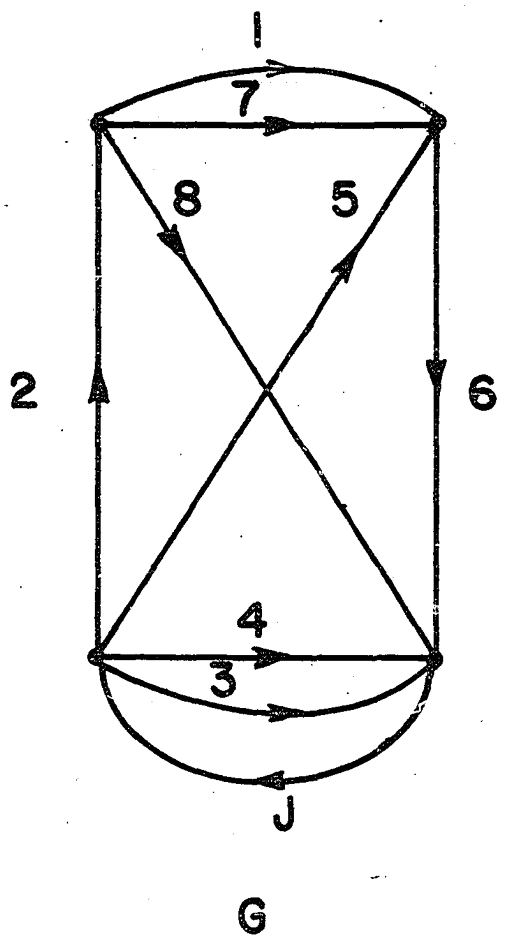


Figure 18. Graph pair for the network of Figure 17

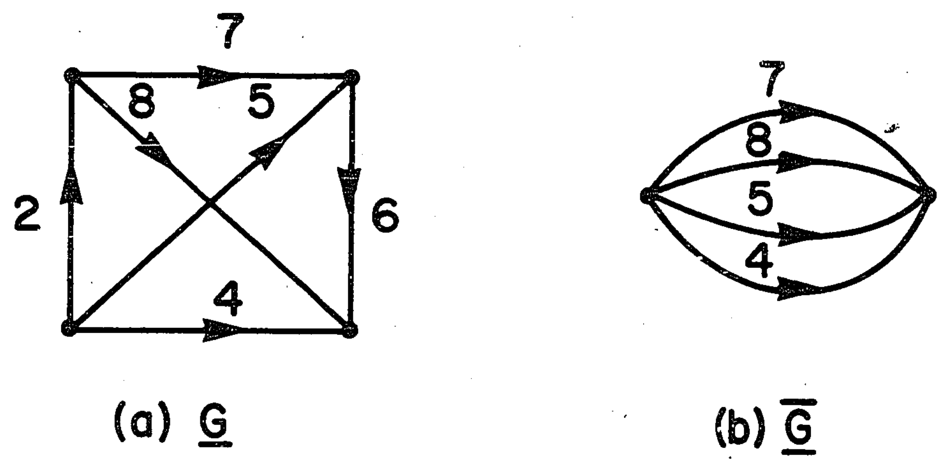


Figure 19. Subgraphs of G used for finding trees t_i

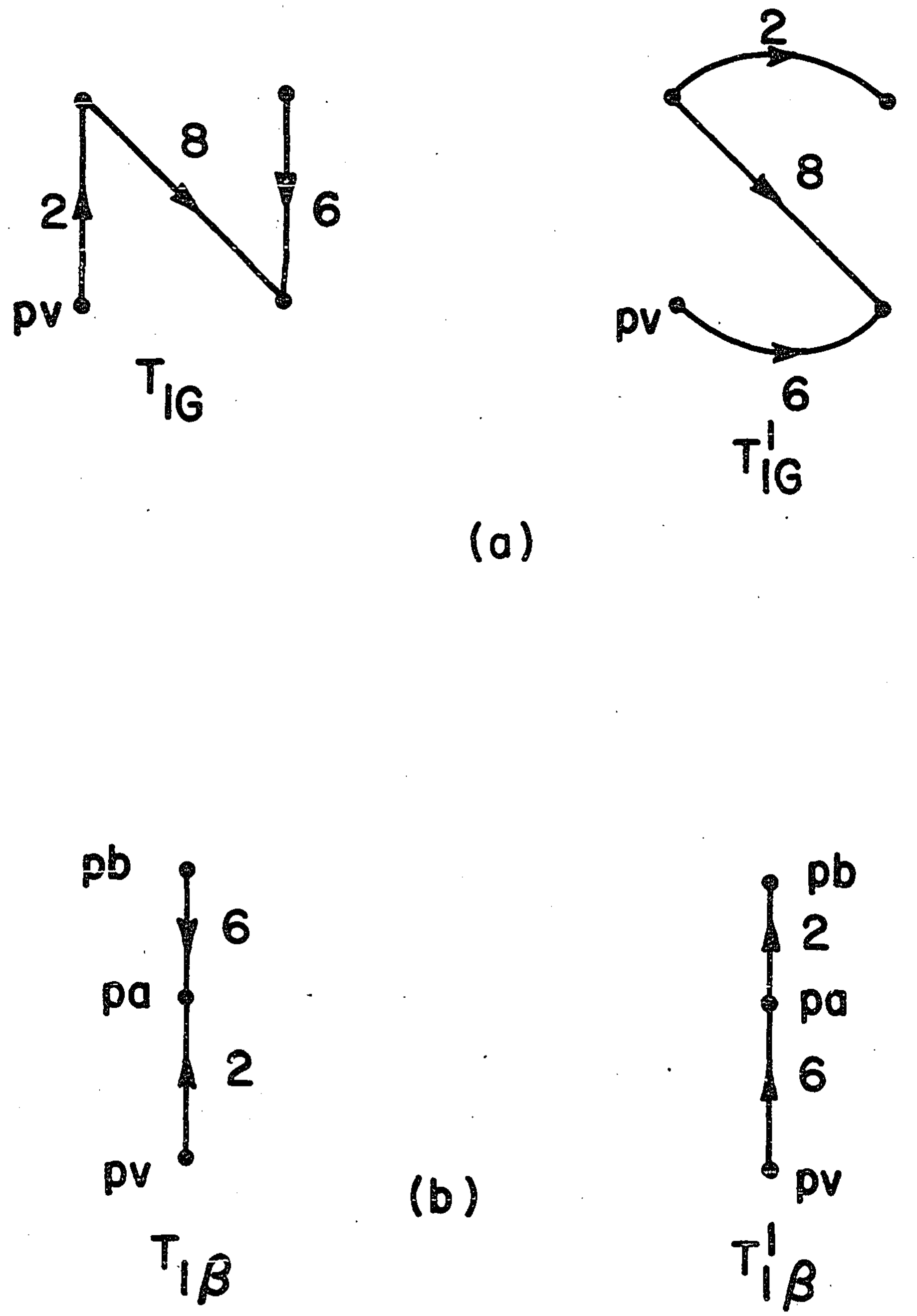
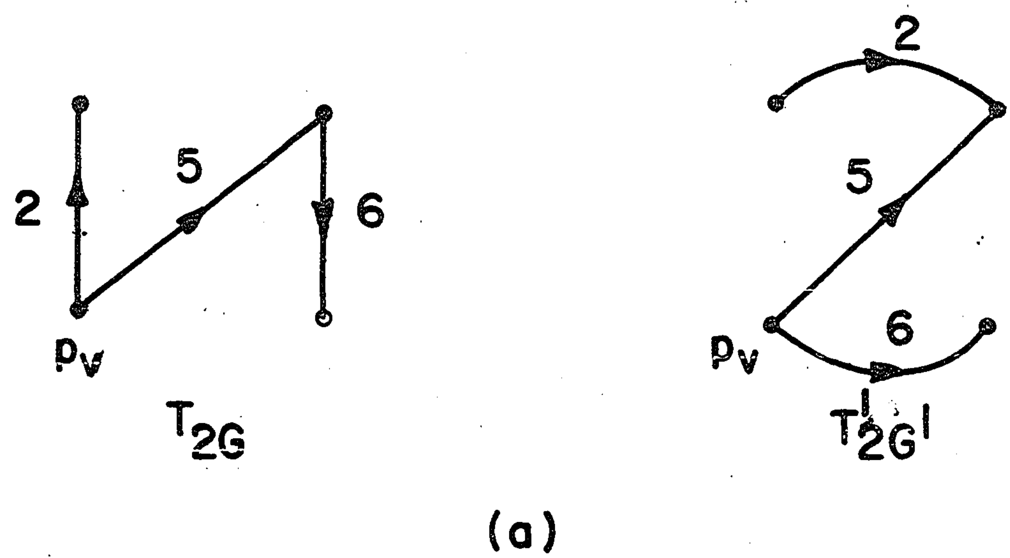


Figure 20. Trees and signatures corresponding to term 268



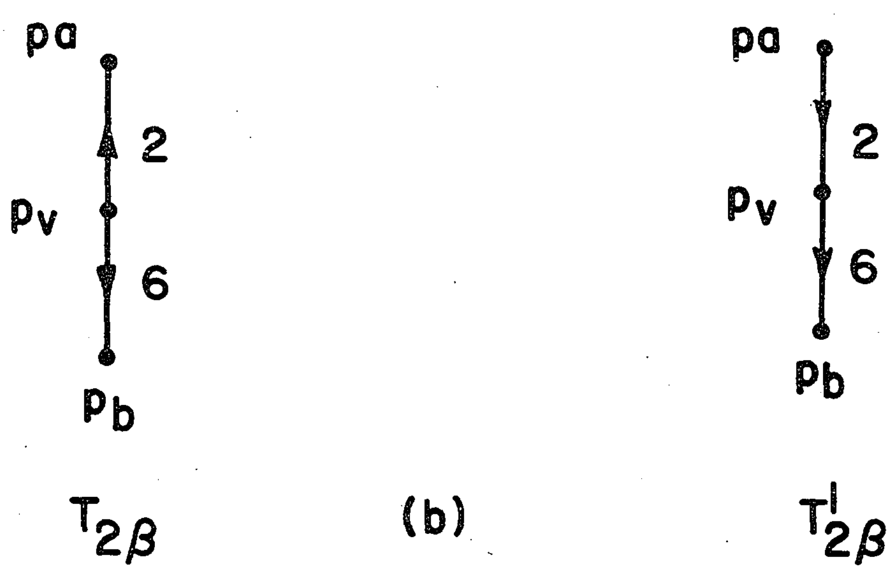


Figure 21. Trees and signatures corresponding to term 265

Since one of the sources of Figure 17 is controlled by the current of admittance 1, $p(Y_p) = y_1$. Combining all of these results, and ignoring the factor $u(k')$ gives

$$\left| \underline{C}_{\infty} \right| = (y_8 - y_5)\, y_1 = y_8\, y_1 - y_5\, y_1. \tag{144}$$

The system determinant calculated for the network of Figure 17 by using the nodal equations is

$$\Delta = \frac{(y_1 + y_7 + y_8)(y_5 + y_4 + y_3)}{\alpha_2\ \alpha_6} + \frac{y_1\ (y_5 + y_4 + y_3)}{\alpha_6} +$$

$$\frac{y_5\ (y_1 + y_7 + y_8)}{\alpha_2} + (y_8\, y_1 - y_5\, y_1). \tag{145}$$

which reduces to Equation 144 in the limit as the transmittance constants approach infinity.

Comparison of Equation 145 with the general expanded determinant of Equation 84 illustrates how the topological formulas derived here fit into the general set of topological formulas which remain to be derived. The general derivation is discussed further in the following chapter.

RESEARCH PROBLEMS SUGGESTED BY THIS STUDY

This investigation has exposed and laid the groundwork for solving a number of new problems which are discussed briefly in this chapter.

(a) In the solvability chapter, it was found that, except for $I\text{-}V_o$ and $V\text{-}I_o$ networks, the unique solvability of a single-source network depends upon whether or not a certain transfer function of a passive 2-port is a real number. Therefore it would be desirable to find a set of conditions both necessary and sufficient to insure that a general passive 2-port has a transfer function independent of the Laplace-transform variable s. The transfer functions studied should include all of the types pertinent to the admittance-controlled, single-source networks and to $I\text{-}I_o$ and $V\text{-}V_o$ networks. Because of the usefulness of topological formulas in problems of this type, it might be expedient to initially limit the study to transformerless 2-ports. A good starting point might be the transfer function which arises in the $I\text{-}I_z$ problem, since each tree sum of the numerator is a sub-set of the tree sum of the denominator in this case.

(b) In each of the solvability theorems for n-source networks, an nth order determinant of the form $\left| F - \mathcal{k}^{-1} \right|$ involving transfer functions and transmittance constants was found to contain the information about the independence of the transmittance constraints and the passive network constraints. It is always possible to express such a determinant in an expanded form similar to that of Equation 44. Furthermore, if the network contains no transformers, it is sometimes possible to express each determinant of the expansion in terms of topological formulas which pertain to the appropriate network N_3. For example, consider the network N_1 shown in Figure 22. If J_i is used to denote the current of the replacement source corresponding to controlled source I_{ci}, the nodal equation for the corresponding network N_2 may be written in the form

$$\begin{bmatrix} (y_a+y_e+y_d) & -y_e & 0 & -y_d \\ -y_e & (y_b+y_f+y_e) & -y_f & 0 \\ 0 & -y_f & (y_c+y_g+y_f) & -y_g \\ -y_d & 0 & -y_g & (y_d+y_g) \end{bmatrix} \begin{bmatrix} v_2 \\ v_3 \\ v_4 \\ v_1 \end{bmatrix} = \begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \end{bmatrix}. \tag{146}$$

The nodal voltages of Equation 146 correspond to the controlling voltages of N_1, and each row on the right side contains the current of a different replacement source. Because of this, the inverse of the coefficient matrix, C, is a 4x4 matrix in which the element of row i and column j is the transfer function, f_{ij}, relating controlling variable i to replacement source j in N_3. That is, in this special case, it happens to be true that $F = C^{-1}$. (N_2 and N_3 are identical in this case since no fixed sources are included in N_1). From Equation 44 it may be seen that each determinant appearing in the expansion of $\left| F - \mathcal{A}^{-1} \right|$ is the determinant of some submatrix of k rows and k columns of F, where the value of k depends upon the number of α's which appear in that particular term in the expansion. Therefore, the array of elements in each such determinant constitutes a submatrix of C^{-1}. However, the inverse of a non-singular matrix is the adjoint matrix divided by the determinant, which gives

$$F = \frac{\text{adj } C}{|C|} \quad . \tag{147}$$

Each submatrix of F is then related to the corresponding submatrix of adj C by the scalar multiplier $\frac{1}{|C|}$. A theorem given by Aitken (1) states that

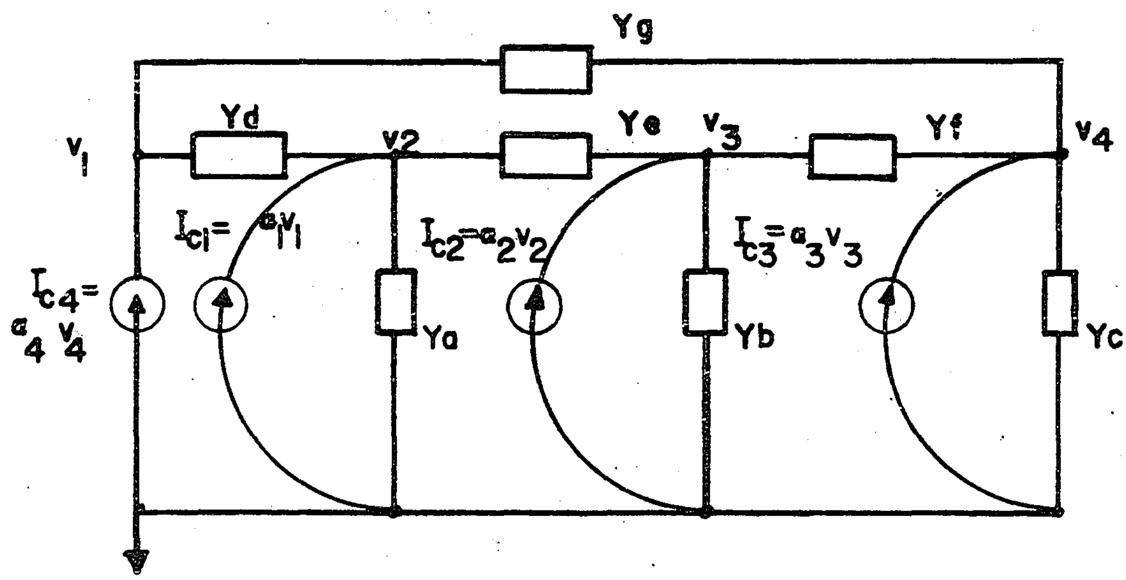


Figure 22. Network illustrating research problem b

any minor of order k in adj C is equal to the complementary signed minor in $\left| C^T \right|$, multiplied by $\left| C \right|^{k-1}$. This means that every determinant in the expansion of $\left| F - \mathcal{A}^{-1} \right|$ may be found from a minor of $\left| C^T \right|$. Since Equation 146 is an ordinary nodal equation,

$$C^T = AYA^T, \tag{148}$$

where A is the vertex matrix for N_3. The minors of $\left| AYA^T \right|$ may easily be found by means of topological formulas, which means that the terms of the expansion of $\left| F - \mathcal{A}^{-1} \right|$ may be given in terms of topological formulas of N_3. This observation suggests that a study be made to determine exactly how general this approach is. That is, what conditions placed upon a controlled-source network are necessary and sufficient to insure that the determinants of the submatrices of F correspond to individual minors of the coefficient matrix of a set of nodal equations?

The equality $F = C^{-1}$, which makes the procedure work in this case, occurs because the controlling variables happen to be nodal variables and because each J from a replacement source appears in only one row on the right side of Equation 146. Thinking in terms of Cramer's rule, these conditions result in each transfer function being proportional to a single cofactor of $|C|$ rather than a difference of cofactors. This same result will also occur if the controlling voltages are only a proper subset of the nodal variables and if some of the rows of the driving function matrix contain zeros.

It should also be possible to apply this method to some classes of networks in which the controlling voltages do not share a common reference node as they do in this example. By using the cut-set equation,

$$QYQ^T V_{np} = J, \qquad (149)$$

a matrix, V_{np}, of node-pair voltage variables is obtained. If the controlling voltages are a subset of elements of V_{np}, and if each replacement source of J appears in a distinct cut-set of Q, the observations made for the special case should also apply here. Finding the topological formulas presents no added difficulties in this case, since minors of QYQ^T may be found by the methods ordinarily used to find minors of AYA^T.

Still more generality may be achieved by investigating equations of the form

$$Q_1 Y Q_2^T V_{np}' = J', \qquad (150)$$

where the cut-set matrix Q_2 determines the node-pair variables of V_{np}', and a different cut-set matrix, Q_1, establishes the form of J'. In this equation, the replacement sources of J' need not be included in the cut-sets which include the controlling elements.

The possibility of finding an analogous procedure involving mesh equations should also be explored.

Finally, since the expanded determinant of Equation 44 is so similar in form to the determinant expansion which results from the direct topological formula studies, for example Equation 84, a comparison of the two expansions might be worth while. The topological formulas derived by the method discussed here will be in terms of trees and sub-trees of N_3, whereas the general derivation method gives results in terms of complete trees of N_3. Therefore, comparison of the two might be useful for interpreting both sets of formulas. Also, the method discussed here will not work for $I-I_o$ or $V-V_o$ sources, since Y does not exist in these cases, whereas the general method will work for these source types. Comparison of the two approaches might give some sort of an intuitive explanation of this. Finally, the close relationship of the formulas derived by this method to the transfer functions suggests that this approach may relate the relatively abstract trees and 2-trees of the topological formulas to the concrete and familiar concepts of transfer functions.

(c) The derivations in the last chapter and the discussion in the appendix demonstrate the feasibility of deriving topological formulas directly from the element-variable equations. Derivation of formulas for system determinants of general controlled-source networks appears to be merely a matter of applying further the techniques already established, although a unifying rule for interpreting the various terms in the general expansion might present difficulties. As mentioned above, the solution to problem (b) might provide insight useful in the interpretation aspect of this problem.

The method should apply equally well to the derivation of topological formulas for transfer functions of controlled source networks. A general expression for the numerator determinant of such a transfer function may be obtained from the system equation by Cramer's rule, after which the numerator may be expanded using the principles developed for the denominator expansion.

(d) Once topological formulas are developed for the determinants of general controlled-source networks, it should be possible to develop similar formulas for further examination of the solvability problem. The networks studied in the solvability chapter differ from those of the last chapter only in containing more than one driving source. The result of this is a larger number of elements in the partitioned coefficient matrix and a corresponding increase in the difficulties involved in interpreting each term of the expanded determinant. The advantage of approaching the solvability problem from this viewpoint is that the same technique is applicable to all network types, even those containing $V\text{-}I_o$ and $I\text{-}V_o$ sources.

SUMMARY

This investigation provides a unified approach to both the solvability problem and the problem of topological formula derivation for networks containing all types of controlled sources. Starting from a network diagram in which short and open-circuits are replaced by suitable fixed sources, a linear graph is drawn which provides a basis for a set of element-variable equations applicable to both problems.

The unique solvability of the system equations which describe most connected, controlled-source networks were found to depend upon two necessary and sufficient conditions. First, uniquely solvable networks may contain neither circuits of voltage sources nor cut-sets of current sources. This purely topological constraint is a simple generalization of the condition required of passive networks which contain fixed driving sources. The second condition, which is unique to controlled-source networks, is that the transmittance constraints imposed between the controlling and controlled variables must be independent of a certain set of constraints imposed between the same variables by a related passive network. The topological location conditions were found to be sufficient but not necessary for the unique solvability of networks containing $I\text{-}V_o$ and $V\text{-}I_o$ sources. However an approach using topological formulas was suggested for further investigation of these networks.

The solvability studies lead to some interesting results for network models in which the transmittance constants approach infinity. It was found that a solvable network equation remains solvable in this limiting case only if no subset of the controlling elements of the network constitute either a seg or a circuit of the network. It was found that a topo-

logical formula presented in the literature to describe operational amplifier networks is a special case of a topological formula derived here to describe the system determinant of a transformerless one-source network.

A theorem was stated and proved which, for a given tree, relates the algebraic signs associated with the determinants of those non-singular submatrices of given vertex and circuit matrices which correspond to that tree. This theorem was used to derive topological formulas for the system determinants of controlled-source networks in which the transmittance constants are allowed to approach infinity. More important, the theorem was shown to provide the key for expanding the determinants of element-variable equations, and thus for deriving topological formulas which apply even to networks for which element-admittance or element-impedance matrices may not be defined.

Implicit in the approach to controlled-source networks presented here is an emphasis on the source properties of the controlled-source elements, constituting a marked departure from conventional approaches which treat these elements as unilateral coupling devices. The advantages of the new viewpoint are greater generality, use of concepts more familiar to electrical engineers, and the ready applicability of the superposition principle for providing insight into new problems.

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APPENDIX

Equation 28 describes a certain controlled-source network N_1. It is known that the system equation for the related network N_2, which contains no controlled source, is given by

$$\begin{bmatrix} A_{p1} & A_{p2} & A_E & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_c & B_{p1} & B_{p2} & B_J \\ -1 & 0 & 0 & 0 & y_{p1} & 0 & 0 \\ 0 & -U & 0 & 0 & 0 & Y_{p2} & 0 \end{bmatrix} \begin{bmatrix} I_{p1} \\ I_{p2} \\ I_E \\ V_c \\ V_{p1} \\ V_{p2} \\ V_J \end{bmatrix} = \begin{bmatrix} -A_J J - A_c J_c \\ -B_E E \\ I_{n1}' \\ I_{n2}' \end{bmatrix}, \qquad (151)$$

where the coefficient matrix is the matrix M' of Equation 29. Our object is to express M' by means of a topological formula. Since the partitioning which separates the controlling admittance, y_{p1}, from the other passive network elements is not necessary here, we may write

$$|M'| = \begin{vmatrix} A_p & A_E & 0 & 0 & 0 \\ 0 & 0 & B_c & B_p & B_J \\ -U & 0 & 0 & Y_p & 0 \end{vmatrix}, \qquad (152)$$

where the subscript p denotes all of the passive elements including y_{p1}. By multiplying some of the rows by -1 and by rearranging the columns, we obtain

$$|M'| = \pm \begin{vmatrix} 0 & 0 & 0 & A_E & A_p \\ B_c & B_p & B_J & 0 & 0 \\ 0 & -Y_p & 0 & 0 & U \end{vmatrix}. \qquad (153)$$

But, by elementary column operations,

$$|M'| = \pm \begin{vmatrix} 0 & A_p Y_p & 0 & A_E & A_p \\ B_c & B_p & B_J & 0 & 0 \\ 0 & 0 & 0 & 0 & U \end{vmatrix} = \pm \begin{vmatrix} 0 & A_p Y_p & 0 & A_E \\ B_c & B_p & B_J & 0 \end{vmatrix} . \tag{154}$$

By the reasoning used in studying the determinant on the right side of Equation 117, Laplace's expansion of Equation 154 according to minors of the first v-1 rows will give a sum of terms from the Laplace's expansion of

$$\pm \begin{vmatrix} 0 & A_p & 0 & A_E \\ B_c & B_p & B_J & 0 \end{vmatrix} = D. \tag{155}$$

Each such term will have a coefficient $p_y(t_j)$, which is the product of the admittances included in a tree t_j corresponding to a nonzero minor of $[A_p \; A_E]$. Note that the terms arising from the expansion of Equation 155 will be a subset of terms of

$$\begin{vmatrix} A_c & A_p & A_J & A_E \\ B_c & B_p & B_J & B_E \end{vmatrix} ;$$

or each nonzero term of Equation 155 will correspond to a tree of N_2. In fact, the expansion of Equation 154 may be written in the form

$$|M'| = \pm \left\{ \sum_{j=1}^{n'} (-1)^{s_{t_j}} \left| A_{t_j} \right| \cdot \left| B_{t_j} \right| p_y(t_j) \right\} , \tag{156}$$

where n' is the number of trees t_j of N_2 which contain every voltage source, but no current source. Using Theorem 19, Equation 156 becomes

$$|M'| = \pm \left\{ u\, k' \left[\sum_{j=1}^{n'} p_y(t_j) \right] \right\}. \tag{157}$$

It is possible to interpret Equation 157 in terms of a topological formula for the network N_3. This may be done by using Theorem 5 if the one-to-one correspondence between the linear graph edges and the elements they represent is kept in mind. As a result of this correspondence, it is possible to associate an operation on the network with each graph operation suggested by the theorem.

Since no tree t_j of N_2 contains a current source, repeated application of Theorem 5 shows that every tree t_j is included in the network N_a formed from N_2 by properly removing all of the current sources. But each tree t_j contains every voltage source of N_2, and therefore of N_a. Repeated application of Theorem 5 shows that the set of trees t_j may be formed by adding all of the voltage source branches to each tree of a new network N_b in a manner which adds no circuits. Therefore, the trees t_j are in one-to-one correspondence with the trees of N_b. But N_b, by Theorem 5, is formed from N_a by properly removing the voltage sources, and therefore contains only admittances. Therefore, each $p_y(t_j)$ corresponds to a tree of N_b; and the summation over j,

$$\sum_{j=1}^{n'} p_y(t_j) \quad ,$$

includes every tree of N_b. But N_b is the network N_3 which corresponds to the given N_2. Therefore,

$$\sum_{j=1}^{n'} p_y(t_j) = \Sigma T \quad , \tag{158}$$

where ΣT is the tree sum for N_3. Substituting Equation 158 into Equation 157 and absorbing the $\pm$ sign into the term u gives

$$M' = u(k') \Sigma T \quad , \tag{159}$$

which is the desired formula.